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ON THE RAY–WALKER EXTENSION OF THE CARISTI–KIRK FIXED POINT THEOREM*

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IN [3], RAY AND WALKER derived mapping theorems for nonlinear operators on Banach spaces satisfying two different kinds of local assumptions—differentiability and monotonicity—by using basic technique involving applications of the following extended version of the well-known Caristi–Kirk fixed point theorem [2].

THEOREM (Ray–Walker [3]). Let (M, d) be a complete metric space, let ψ be a lower semicontinuous function from M to $[0, \infty)$, let c be a continuous nonincreasing function from $[0, \infty)$ to $[0, \infty)$, for which $\int^{\infty} c(s)\,ds = \infty$, and let $x_0 \in M$ fixed. If $g: M \to M$ is a mapping satisfying

$$c(d(x, x_0))d(x, g(x)) \leqslant \psi(x) - \psi(g(x)), \qquad x \in M,$$

then g has a fixed point.

If $c(s) \equiv 1$, then the Ray–Walker theorem reduces to the Caristi–Kirk theorem, which is equivalent to Ekeland's theorem [1].

In this short paper, we show that the Ray–Walker theorem is actually equivalent to the Caristi–Kirk theorem.

The following lemma shows our claim.

LEMMA. Let (M, d) be a metric space, let $\psi: M \to [0, \infty)$ be a function, let $c: [0, \infty) \to [0, \infty)$ be a continuous nonincreasing function with $\int^{\infty} c(s)\,ds = \infty$, and let $x_0 \in M$. Then there exists a function $\phi: M \to [0, \infty)$ satisfying

$$d(x, y) \leqslant \phi(x) - \phi(y)$$

whenever

$$c(d(x, x_0))d(x, y) \leqslant \psi(x) - \psi(y).$$

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Moreover, if ψ is l.s.c., so is ϕ.

Proof. Since c is nonincreasing, continuous and $\int^{\infty} c(s)\,\mathrm{d}s = \infty$, we can define $\phi: X \to [0, \infty)$ by

$$\int_{d(x,x_0)}^{d(x,x_0)+\phi(x)} c(s)\,\mathrm{d}s = \psi(x).$$

Moreover, if ψ is l.s.c. at $x \in M$ and if $x_n \to x$, then

$$\underline{\lim}\, \psi(x_n) = \underline{\lim} \int_{d(x_n,x_0)}^{d(x_n,x_0)+\phi(x_n)} c(s)\,\mathrm{d}s \geqslant \psi(x).$$

Since $\lim d(x_n, x_0) = d(x, x_0)$, we have $\underline{\lim}\, \phi(x_n) \geqslant \phi(x)$. Hence, ϕ is also l.s.c. at x. Now suppose that

$$c(d(x, x_0))d(x, y) \leqslant \psi(x) - \psi(y).$$

Since c is nonincreasing,

$$\int_{d(x,x_0)}^{d(x,x_0)+d(x,y)} c(s)\,\mathrm{d}s \leqslant c(d(x, x_0))d(x, y).$$

Therefore, we have

$$\int_{d(x,x_0)}^{d(x,x_0)+d(x,y)} c(s)\,\mathrm{d}s \leqslant \int_{d(x,x_0)}^{d(x,x_0)+\phi(x)} c(s)\,\mathrm{d}s - \int_{d(y,x_0)}^{d(y,x_0)+\phi(y)} c(s)\,\mathrm{d}s.$$

Since $d(y, x_0) \leqslant d(x_0, x) + d(x, y)$ and c is nonincreasing,

$$\int_{d(x,x_0)}^{d(x,x_0)+d(x,y)} c(s)\,\mathrm{d}s + \int_{d(x,x_0)+d(x,y)}^{d(x,x_0)+d(x,y)+\phi(y)} c(s)\,\mathrm{d}s \leqslant \int_{d(x,x_0)}^{d(x,x_0)+\phi(x)} c(s)\,\mathrm{d}s,$$

which shows that

$$d(x, y) + \phi(y) \leqslant \phi(x).$$

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