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REMARKS ON SURJECTIVITY OF ϕ-ACCRETIVE OPERATORS

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In [2] and [3], the authors obtained surjectivity results on generalized locally ϕ-accretive operators. In the present paper, we note that strongly upper semicontinuous map defined in [2] is single-valued and continuous, and restate the results in [2], [3] more accurately. Further, we raise open problems on the duality map J of a Banach space.

Let us define the duality map J from a Banach space X into 2^{X^*} as follows:

$$J(x)=\{x^*\in X^* \mid \langle x,\ x^*\rangle=\|x^*\|^2=\|x\|^2\}$$

for $x\in X$, where X^* is the dual of X. By the Hahn-Banach theorem, $J(x)\neq\phi$ for all $x\in X$.

Let Y be a Banach space and J its duality map. In [2], J is said to be *strongly upper semicontinuous* if the following condition holds:

(1) if $\lim_{n\to\infty} y_n=y$, $y_n^*\in J(y_n)$, and $y^*\in J(y)$, then y^* is a subsequential (strong) limit of $\{y_n^*\}$.

We note that such J is single-valued and continuous. For, if $\lim_{n\to\infty} y_n=y$ and $y_n^*\in J(y_n)$, then for any $y^*\in J(y)$, we have $\lim_{n\to\infty} y_n^*=y^*$. Otherwise, we can find $\varepsilon>0$ and $\{y_{n_k}^*\}$ such that $\|y_{n_k}^*-y^*\|\geq\varepsilon$. Since $y_{n_k}^*\in J(y_{n_k})$, $y_{n_k}\to y$, and $y^*\in J(y)$, y^* is a subsequential limit of $\{y_{n_k}^*\}$, a contradiction. Hence, $J(y)$ is single.

The example of a strongly upper semicontinuous single-valued map F which is not continuous in [2] is incorrect.

The duality map J is said to be *lower semicontinuous* if the following condition holds:

(2) if $\lim_{n\to\infty} y_n=y$ and $y^*\in J(y)$, then there exists a sequence $\{y_n^*\}$ such that $y_n^*\in J(y_n)$ and $\lim_{n\to\infty} y_n^*=y_*$.

Let X and Y be Banach spaces and $\phi: X\to Y^*$ a map satisfying the following:

(3) $\phi(X)$ is dense in Y^*, and

(4) for each $x\in X$ and each $\alpha>0$, $\|\phi(x)\|\leq\|x\|$ and $\phi(\alpha x)=\alpha\phi(x)$.

A map $P: X\to Y$ is said to be *locally strongly ϕ-accretive* [1] if for each $y\in Y$ and $r>0$ there exists a constant $c>0$ such that

(5) if $\|Px-y\|\leq r$, then, for all $u\in X$ sufficiently near to x,

$$\langle Pu-Px,\ \phi(u-x)\rangle\geq c\|u-x\|^2.$$

Moreover, a map $P: X\to Y$ is said to be *generalized locally ϕ-accretive* [3] if for each $y \in Y$ and $r > 0$ there exists a nonincreasing function c: $[0, \infty) \to (0, \infty)$ such that

(6) if $\|Px-y\| \leq r$, then, for all $u \in X$ sufficiently near to x,

$$<Pu-Px, \ \phi(u-x)> \geq c(\|x\|)\|u-x\|^2.$$

Note that (5) implies (6), and not conversely.

Now our main result in [2] can be restated as follows:

THEOREM 1. *Let X and Y be Banach sapces and P: $X \to Y$ a locally Lipschitzian and locally strongly ϕ-accretive map.*

(i) *If the duality map J of Y is l.s.c., then $P(X)$ is open.*

(ii) *Further, if $P(X)$ is closed, then P is surjective.*

Note that slight modification of the proof of Theorem 2 of [2] works for Theorem 1. The following is obtained in [3], as a generalization of Theorem 1.

THEOREM 2. *Let X and Y be Banach spaces and P: $X \to Y$ a locally Lipschitzian and generalized locally ϕ-accretive map.*

(i) *If the duality map J of Y is l.s.c., then $P(X)$ is open.*

(ii) *Further, if $P(X)$ is closed, then P is surjective.*

Finally, we raise open problems in regard to the above remarks:

(a) Is there any concrete Banach space such that its duality map is *not* single-valued and l.s.c.?

(b) Is there any necessary and sufficient condition on the norm of a Banach space in order that J is *not* single-valued and l.s.c.?

It is well-known that J is single-valued iff the norm is Gâteaux differentiable.

References

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