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REMARKS ON FIXED POINT THEOREMS OF DOWNING AND KIRK FOR SET-VALUED MAPPINGS IN METRIC AND BANACH SPACES

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1. Introduction

In [2], D. Downing and W. A. Kirk obtained a number of fixed point theorems for set-valued maps in metric and Banach spaces. The authors considered maps which are more general than the contractions with nonempty and closed mapping values, and obtain results for maps satisfying certain "inwardness" conditions. A key aspect of their approach is the application of a general fixed point theorem due to Caristi [1].

On the other hand, in [6], the present author obtained a number of equivalent formulations of the well-known result of I. Ekeland [3, 4] on the variational principle for approximate solutions of minimization problems. Some of such formulations include sharpened forms of the Caristi theorem. In this paper, using one of such formulations, we show that Theorems 1-3 and Corollaries 1-5 of [2] are substantially improved by giving geometric estimations of fixed points.

The key to our approach is the applications of the following.

THEOREM 0 [6]. *Let V be a metric space, C a nonempty complete subset of V, $u \in C$, $\varepsilon > 0$ and $\lambda > 0$. Suppose there exists a l.s.c. function $F : C \to \mathbf{R} \cup \{+\infty\}$, $\not\equiv +\infty$, bounded from below such that $F(u) \leq \inf_C F + \varepsilon$.*

If $T : C \to 2^V$ is a set-valued map satisfying the condition:

$\forall x \in \bar{B}(u, \lambda) \cap C/Tx \quad \exists y \in C/\{x\}$ *such that*

$$F(y) \leq F(x) - \varepsilon\lambda^{-1} d(x, y),$$

then T has a fixed point $v \in \bar{B}(u, \lambda) \cap C$ such that $F(v) \leq F(u)$.

In Theorem 0, 2^V denotes the class of all nonempty subsets of V, and $\bar{B}$ the closed ball. Note that v is a fixed point of T iff $v \in Tv$.

We follow terminologies and notations in [2].

For historical remarks, see [2].

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2. Basic results

Let X be a Banach space, $K \subset X$, and $\Gamma : K \to F(X)$ where $F(X)$ denotes the class of nonempty closed subsets of X. Given $x \in K$ and $\alpha \geq 1$, we let

$$\Gamma_\alpha(x) = \{ z \in \Gamma x : \|x - z\| \leq \alpha\, d(x, \Gamma x) \}$$

where $d(x, \Gamma x) = \inf\{ \|x - u\| : u \in \Gamma x \}$. For $x \in K$, the *inward set* of x relative to K is the set

$$I_K(x) = \{ (1-\alpha)x + \alpha y : y \in K, \ \alpha \geq 0 \}.$$

THEOREM 1. *Let K be a nonempty closed convex subset of a Banach space X, and $\Gamma : K \to F(X)$ an u.s.c. map which satisfies for given $k \in (0, 1)$:*

(a) $\forall x \in K \ \ \exists \delta = \delta(x) > 0$ *such that*

$$y \in \bar{B}(x, \delta) \cap K \ \Rightarrow \ d(y, \Gamma y) \leq d(y, \Gamma x) + k \|x - y\|.$$

If either

(b) $\Gamma_1(x) \cap I_K(x) \neq \phi$ *for each* $x \in K$, *or*

(b′) $\forall x \in K \ \ \exists \alpha = \alpha(x) > 1 \ \ \exists \mu = \mu(x) \in (0, 1)$ *such that*

$$(1 - \mu)x + \mu \Gamma_\alpha(x) \subset K,$$

then there exists an $\eta \in (0, 1)$ such that for any $u \in K$ and $\lambda > 0$ satisfying $d(u, \Gamma u) \leq \lambda\eta$, Γ has a fixed point in $\bar{B}(u, \lambda) \cap K$.

Proof. (b) As in the proof of [2, Theorem 1(b)], for any $x \in K$ such that $d(x, \Gamma x) > 0$, there exists a $y \in K / \{x\}$ such that

$$\eta\|x - y\| \leq d(x, \Gamma x) - d(y, \Gamma y),$$

where η is so chosen that $\eta = -[k - (1 - \varepsilon)(1 + \varepsilon)^{-1}]$ for any $\varepsilon > 0$ so that $k < (1 - \varepsilon)(1 + \varepsilon)^{-1}$. Let $F(x) = \eta^{-1} d(x, \Gamma x)$. Then F is l. s. c. since Γ is u. s. c. Therefore by Theorem 0, for any $u \in K$ and $\lambda > 0$ satisfying $F(u) \leq \inf_K F + \lambda$, Γ has a fixed point $v \in \bar{B}(u, \lambda) \cap K$. Since $\inf_K F = 0$, u can be so chosen that $d(u, \Gamma u) \leq \lambda\eta$.

(b′) As in the proof of [2, Theorem 1(b′)], for any $x \in K$ such that $d(x, \Gamma x) > 0$, there exists a $y \in K / \{x\}$ such that

$$\eta\|x - y\| \leq d(x, \Gamma x) - d(y, \Gamma y)$$

where $\eta = 1 - k'$ for some $k' \in (k, 1)$. Therefore, the same argument to the proof (b) leads the conclusion. This completes our proof.

Let $F_b(X)$ denote the class of nonempty bounded closed subsets of a metric space (X, d) with the Hausdorff metric H. A map $\Gamma : K \to F_b(X)$, $K \subset X$, is called a *contraction* if there exists a constant $k \in (0, 1)$ such that

$$H(\Gamma x, \Gamma y) \leq k\, d(x, y), \quad x, y \in K.$$

If the map Γ in Theorem 1 is a contraction from K into $F_b(X)$, then Γ is u. s. c. and satisfies the condition (a), for

$$d(y, \Gamma y) \leq d(y, \Gamma x) + H(\Gamma x, \Gamma y), \quad x, y \in K.$$

In this case, using an argument of T. E. Williamson, Jr. [7], we have more stronger conclusion.

COROLLARY 1. *With K and X as in* Theorem 1, *suppose $\Gamma : K \to F_b(X)$ is a contraction which satisfies either condition* (b) *or* (b′). *Then there exists* an $\eta \in (0,1)$ *such that for any $u \in K$ and $\lambda > 0$ satisfying $d(u, \Gamma u) \leq \lambda\eta$, either u is a fixed point of Γ or Γ has a fixed point v in $\bar{B}(u,\lambda) \cap K/B(u,s)$ where $s = d(u, \Gamma u)(1+k)^{-1}$.*

Proof. Note that $s < \lambda$, for $\eta \in (0,1)$ and

$$s = \frac{d(u, \Gamma u)}{1+k} \leq \frac{\lambda\eta}{1+k} < \lambda.$$

Now it suffices to show that if u is not fixed under Γ, then Γ has no fixed point in $B(u,s)$. For any $y \in B(u,s) \cap K$, we have

$$\begin{aligned} d(u, \Gamma u) &\leq d(u, y) + d(y, \Gamma u) \\ &< s + d(y, \Gamma u), \end{aligned}$$

that is,

$$k(1+k)^{-1} d(u, \Gamma u) < d(y, \Gamma u).$$

Hence,

$$d(y, \Gamma u) > ks > k\, d(y, u).$$

Suppose $y \in \Gamma y$. Then we have

$$H(\Gamma y, \Gamma u) > k\, d(y,u),$$

a contradiction. This completes our proof.

Let $K(X)$ and $H(X)$ be the class of nonempty compact subsets and nonempty weakly compact subsets of X, respectively. Now, we have the following consequence of Corollary 1.

COROLLARY 2. *With K and X as in* Theorem 1, *suppose $\Gamma : K \to F_b(X)$ is a contraction for which $\Gamma x \subset \overline{I_K(x)}$, $x \in K$. If $\Gamma : K \to K(X)$ or if X is reflexive and $\Gamma : K \to H(X)$, there exists an $\eta \in (0,1)$ such that for any $u \in K$ and $\lambda > 0$ satisfying $d(u, \Gamma u) \leq \lambda\eta$, either $u \in \Gamma u$ or Γ has a fixed point v in $\bar{B}(u, \lambda) \cap K/B(u,s)$ where $s = d(u, \Gamma u)(1+k)^{-1}$.*

Recall that a map $f : K \to X$ is *weakly inward* [5] if $fx \in \overline{I_K(X)}$ for each $x \in K$.

COROLLARY 3. *Let K and X be as in* Theorem 1. *Suppose a map $f : K \to X$ is continuous, weakly inward and satisfies for given $k \in (0,1)$:*

(a′) $\forall x \in K \ \exists \delta = \delta(x) > 0$ *such that*

$$y \in \bar{B}(x, \delta) \cap K \ \Rightarrow \ \|x - fy\| \leq \|y - fx\| + k\|x - y\|.$$

Then there exists an $\eta \in (0,1)$ such that for any $u \in K$ and $\lambda > 0$ satisfying $d(u, fu) \leq \lambda\eta$, f has a fixed point in $\bar{B}(u, \lambda) \cap K$.

3. Metric spaces

Let $F(M)$ denote the class of nonempty closed subsets of a metric space (M, d).

THEOREM 2. *Let M be a complete metric space and $\Gamma : M \to F(M)$ a map such that the map $x \mapsto d(x, \Gamma x)$ is l.s.c. Suppose there exist constants $\alpha \geq 1$ and $k < 1$ such that for each $x \in M$,*

$$\inf_{y \in \Gamma_\alpha(x)} d(y, \Gamma y) \leq k\, d(x, \Gamma x) \tag{1}$$

where $\Gamma_\alpha(x) = \{z \in \Gamma x : d(x, z) \leq \alpha\, d(x, \Gamma x)\}$. Then for any $u \in M$, $\eta > 1$, and $\varepsilon > 0$ satisfying $d(u, \Gamma u) \leq \varepsilon(1-k)\alpha^{-1}\eta^{-1}$, Γ has a fixed point in $\bar{B}(u, \varepsilon)$.

Proof. Let $x \in M$, $y \in \Gamma_\alpha(x)$. Then from (1), we have

$$\begin{aligned} \alpha^{-1} d(x, y) &\leq d(x, \Gamma x) \\ &\leq (1-k)^{-1}[d(x, \Gamma x) - \inf_{y \in \Gamma_\alpha(x)} d(y, \Gamma y)]. \end{aligned} \tag{2}$$

For any $x \in M$ such that $x \notin \Gamma x$, we have

$$d(x, \Gamma x) - \inf_{y \in \Gamma_\alpha(x)} d(y, \Gamma y) > 0.$$

Then, for any $\eta > 1$, use (2) to select $y \in \Gamma_\alpha(x)$ so that

$$d(x, \Gamma x) \leq \eta(1-k)^{-1}\, [d(x, \Gamma x) - d(y, \Gamma y)].$$

Let $F(x) = \alpha\eta(1-k)^{-1} d(x, \Gamma x)$. Since F is l.s.c. and $y \neq x$, by Theorem 0, for any $u \in M$ and any $\varepsilon > 0$ such that $F(u) \leq \inf_M F + \varepsilon$, T has a fixed point $v \in \bar{B}(u, \varepsilon)$. Since $\inf_M F = 0$, u can be so chosen that $d(u, \Gamma u) \leq \varepsilon(1-k)\alpha^{-1}\eta^{-1}$. This completes the proof.

COROLLARY 4. *Let M be a complete metric space and $\{f_\gamma\}_{\gamma \in A}$ a pointwise equicontinuous semigroup of selfmaps of M. Suppose there exist constants $\alpha \geq 1$, $k < 1$ such that for any $x \in M$,*

$$\text{(c)} \qquad \begin{aligned} &\sup_{\gamma \in A} d(x, f_\gamma x) \leq \alpha \inf_{\gamma \in A} d(x, f_\gamma x), \\ &\inf_{\mu \neq \gamma} d(f_\mu x, f_\gamma x) \leq k \inf_{\gamma \in A} d(x, f_\gamma x). \end{aligned}$$

Then for any $u \in M$, $\eta > 1$, and $\varepsilon > 0$ satisfying

$$\inf_{\gamma \in A} d(u, f_\gamma u) \leq \varepsilon(1-k)\alpha^{-1}\eta^{-1},$$

there is a point $v \in \bar{B}(u, \varepsilon)$ such that $v = f_\gamma v$ for all $\gamma \in A$.

Proof. For $x \in M$, let $O(x) = \{f_\gamma(x) : \gamma \in A\}$ and define $\Gamma : M \to 2^M$ by taking $\Gamma x = \overline{O(x)}$. Then as in the proof of [2, Corollary 4], $x \longmapsto d(x, \Gamma x)$ is l.s.c., $\Gamma_\alpha(x) = \Gamma x$, and

$$\inf_{y \in \Gamma_\alpha(x)} d(y, \Gamma y) \leq k\, d(x, \Gamma x).$$

Therefore, for any $u \in M$, $\eta > 1$, and $\varepsilon > 0$ satisfying $d(u, \Gamma u) \leq \varepsilon(1-k)\alpha^{-1}$

η^{-1}, Γ has a fixed point $v \in \bar{B}(u, \varepsilon)$. Note that by the first inequality of (c), $v \in \Gamma v$ can happen only if $v = f_\gamma(v)$ for all $\gamma \in A$. This completes the proof.

From Corollary 4, we have the following.

COROLLARY 5. *Let M be a complete metric space and $f: M \to M$ a map for which $\{f^i\}_{i=1}^\infty$ is pointwise equicontinuous on M. Suppose for fixed $\alpha \geq 1$ and $k < 1$, f satisfies the condition*

$$(\mathrm{c}') \qquad \begin{aligned} &\sup_{i\geq 1} d(x, f^i x) \leq \alpha \inf_{i\geq 1} d(x, f^i x), \\ &\inf_{j>i\geq 1} d(f^i x, f^j x) \leq k \inf_{i\geq 1} d(x, f^i x), \quad x \in M. \end{aligned}$$

Then for any $u \in M$, $\eta > 1$, and $\varepsilon > 0$ satisfying

$$\inf_{i\geq 1} d(u, f^i u) \leq \varepsilon(1-k)\alpha^{-1}\eta^{-1},$$

f has a fixed point $v \in \bar{B}(u, \varepsilon)$.

REMARK. If $f: M \to M$ is a contraction with Lipschitz constant $k < 1$, then f satisfies the hypothesis of Corollary 5 with $\alpha = (1-k)^{-1}$. Therefore, as in Corollary 1, for any $u \in M$, $\eta > 1$, and $\varepsilon > 0$ satisfying

$$\inf_{i\geq 1} d(u, f^i u) \leq \varepsilon\eta^{-1},$$

either $u = fu$ or f has a fixed point $v \in \bar{B}(u, \varepsilon)/B(u, s)$ where $s = (1+k)^{-1} \inf_{i\geq 1} d(u, f^i u)$. This improves the Banach contraction principle.

Recall that a metric space is *convex* iff for each two points x, y, $x \neq y$, there exists a point z, $x \neq z \neq y$, such that

$$d(x, z) + d(z, y) = d(x, y).$$

THEOREM 3. *Let M be a complete convex metric space and $f: M \to M$ a surjection which satisfies, for fixed $h > 1$, the condition:*

$$(\mathrm{d}) \qquad \begin{aligned} &\forall x \in M \ \ \exists \varepsilon = \varepsilon(x) > 0 \ \ \textit{such that} \\ &d(x, y) \leq \varepsilon \ \Rightarrow \ d(fx, fy) \geq h\, d(x, y). \end{aligned}$$

If $f^{-1}: M \to 2^M$ is continuous with $f^{-1}x$ compact for each $x \in M$, then for any $u \in M$ and $\varepsilon > 0$ satisfying $d(u, f^{-1}u) \leq (1-h^{-1})\lambda$, f has a fixed point $v \in \bar{B}(u, \lambda)$.

Proof. Define $F: M \to \mathbf{R}$ by $F(x) = d(x, f^{-1}x)$. Then for any $x \in M$ such that $x \neq fx$, as in the proof of [2, Theorem 3], there exists an $m \in M$ such that $m \neq x$ and

$$d(x, m) \leq (1-h^{-1})^{-1}[F(x) - F(m)].$$

Thus by Theorem 0, for any $u \in M$ and $\lambda > 0$ satisfying $F(u) \leq \inf_M F + (1-h^{-1})\lambda$, f has a fixed point $v \in \bar{B}(u, \lambda)$ such that $F(v) \leq F(u)$. In fact, since $\inf_M F = 0$, u can be so chosen that $d(u, f^{-1}u) \leq (1-h^{-1})\lambda$.

REMARK. Each of Theorems 1–3 and Corollaries 1–5 improves the corresponding one in [2] in the sense that ours give the geometric estimations of whereabouts of fixed points. Note that we have relied heavily upon basic ideas of Downing and Kirk.

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