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EXTENSIONS OF THE WEAK CONTRACTIONS OF DUGUNDJI AND GRANAS

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1. Introduction

A Banach contraction is a selfmap f of a metric space (X, d) satisfying $d(fx, fy) \leq \alpha\, d(x, y)$ for all $x, y \in X$ and for some $\alpha \in [0, 1)$. The well-known Banach contraction principle states that for complete X, such f has a unique fixed point p and $f^n x \rightarrow p$ for all $x \in X$. There have been numerous literatures on extensions of the principle.

In [3], J. Dugundji and A. Granas extend the principle to a contractive type map which is called a weak contraction and obtain some applications including a domain invariance theorem. They also introduce the concept of weakly expansive maps and establish some of their properties.

In the present paper, we show that the Dugundji–Granas contraction is actually particular to some well known other contractive type maps and that their fixed point results are actually particular to those in [6], [7], [8], and [9]. Moreover, following the methods in [7], [8], [9], we show that fixed point theorems on weak contractions and on weakly expansive maps can be unified to a single theorem. Furthermore, we extend the domain invariance theorem for weakly contractive fields to the one for Meir–Keeler type contractive fields. In the sequel, we follow the notations of [7].

2. Comparisons of contractive type conditions

Let R_+ denote the set of nonnegative reals. A map $\phi : R_+ \rightarrow R_+$ is said to be compactly positive if $\inf\{\phi(t) \mid a \leq t \leq b\} = \lambda(a, b) > 0$ for any $b > a > 0$ [3].

Consider the following conditions on a selfmap f of a metric space

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(X, d):

(Dd) There exists an increasing right continuous function $\phi : R_+ \to R_+$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$, we have

$$d(fx, fy) \leq \phi(d(x, y)).$$

(DG) There exists a compactly positive function $\phi : R_+ \to R_+$ such that for any $x, y \in X$,

$$d(fx, fy) \leq d(x, y) - \phi(d(x, y)).$$

(Cd) Given $\varepsilon > 0$, there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta_0 \quad \text{implies} \quad d(fx, fy) \leq \varepsilon_0.$$

(Bd) Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(fx, fy) < \varepsilon.$$

Note that the condition (Dd) is due to Browder [2], (DG) to Dugundji–Granas [3], (Cd) to Boyd–Wong [1] (also see Hegedüs–Szilagyi [4]), and (Bd) to Meir–Keeler [6].

The following is basic.

LEMMA. (Dd) $\Longrightarrow$ (DG) $\Longrightarrow$ (Cd) $\Longrightarrow$ (Bd).

Proof. (Dd) $\Longrightarrow$ (DG) is given as Proposition (3.2) by Dugundji–Granas [3]. (Cd) $\Longrightarrow$ (Bd) is given by Meir–Keeler [6]. It remains to show that (DG) $\Longrightarrow$ (Cd). In [3], it is shown that (DG) is equivalent to the following condition of Krasnoselskij [5]:

(1) There exists a map $\alpha : R_+ \to R_+$ satisfying $\sup\{\alpha(t) \mid a \leq t \leq b\} < 1$ for any $b > a > 0$, such that for any $x, y \in X$,

$$d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y).$$

On the other hand, in [4], it is shown that (Cd) is equivalent to the following:

(2) There exists a map $\alpha : R_+ \to [0, 1)$ such that for any $\varepsilon > 0$ there exists a $\delta > 0$ with $\sup\{\alpha(t) \mid \varepsilon \leq t < \varepsilon + \delta\} < 1$ and, for any $x, y \in X$,

$$d(fx, fy) \leq \alpha(d(x, y)) \cdot d(x, y).$$

Then it is clear that (1) $\Longrightarrow$ (2). This completes our proof.

REMARK. An example showing (Bd)$\not\Longrightarrow$(Cd) is given by Meir-Keeler [6].

3. Fixed point theorems

Let f be a continuous selfmap of a metric space (X, d), $C_f=\{g: X \to X \mid fg=gf,\ gX\subset fX\}$. For $x_0\in X$ the sequence $\{fx_n\}_{n=1}^{\infty}$ is called the f-iteration of x_0 under g, as defined by $fx_n=gx_{n-1}$, $n=0,1,2,\ldots$, with the understanding that, if $fx_n=fx_{n+1}$ for some n, then $fx_{n+j}=fx_n$ for each $j\geq 0$. The set $\{fx_n\}_{n=1}^{\infty}$ will be denoted by $O(x_0)$. A point $x_0\in X$ is said to be regular if $\operatorname{diam} O(x_0)<\infty$.

The following is a consequence of Theorem 2(Cδ)$'$ in [7].

THEOREM 3.1. *Let f be a continuous selfmap of a complete metric space (X, d) and $g\in C_f$ continuous. Suppose that X contains a regular point and that*

(Cδ)$'$ *for any $\varepsilon>0$, there exist $\varepsilon_0<\varepsilon$ and $\delta_0>0$ such that for any regular points $x, y\in X$,*

$$\varepsilon\leq \operatorname{diam}\ (O(x)\cup O(y))<\varepsilon+\delta_0 \text{ implies } d(gx, gy)\leq\varepsilon_0.$$

Then f and g have a unique common fixed point p in X, and, for any regular $x_0\in X$, any f-iteration of x_0 under g converges to some $\xi\in X$ satisfying $f\xi=p$.

THEOREM 3.2. [9, Theorem 4] *Let f be a continuous selfmap of a complete metric space (X, d), $g\in C_f$ continuous and f, g satisfying the following condition:*

(Bk)$'$ *For each $\varepsilon>0$ there exists a $\delta>0$ such that $\varepsilon\leq\max\{d(fx, fy), d(fx, gx), d(fy, gy), [d(fx, gy)+d(fy, gx)]/2\}<\varepsilon+\delta$ implies $d(gx, gy)<\varepsilon$.*

Then f and g have a unique common fixed point p in X, and, for any $x_0\in X$, any f-iteration of x_0 under g converges to some $\xi\in X$ satisfying $f\xi=p$.

If $f=1_X$, the condition (Bk)$'$ will be denoted by (Bk) [10]. Now, from Theorem 3.2, we have

THEOREM 3.3. [8, Theorem 2.4] *Let f be a continuous selfmap of a complete metric space X, and $g\in C_f$, satisfying*

(Bd)′ *for each* $\varepsilon>0$ *there exists a* $\delta>0$ *such that, for all* $x, y\in X$,

$$\varepsilon \leq d(fx,fy) < \varepsilon+\delta \text{ implies } d(gx, gy) < \varepsilon.$$

Then f *and* g *have a unique common fixed point* p *in* X, *and, for any* $x_0\in X$, *any* f*-iteration of* x_0 *under* g *converges to some* $\xi\in X$ *satisfying* $f\xi=p$.

Now consider the following condition on f and g:

(DG)′ There exists a compactly positive function $\phi : R_+ \to R_+$ such that for any $x, y \in X$, $d(gx, gy) \leq d(fx, fy) - \phi(d(fx, fy))$.

Imitating (DG)$\Longrightarrow$(Bd)$\Longrightarrow$(Bk) and (DG)$\Longrightarrow$(Cd)$\Longrightarrow$(Cδ) as in Section 2 and in [4], [10], we have (DG)′$\Longrightarrow$(Bk)′ and (DG)′ $\Longrightarrow$(Cδ)′.

Therefore, from Theorems 3.1 and 3.3 we obtain the following

THEOREM 3.4. *Let* f *be a continuous selfmap of a complete metric space* X *and* $g\in C_f$ *satisfying the condition* (DG)′. *Then* f *and* g *have a unique common fixed point* p *in* X, *and, for any* $x_0 \in X$, *any* f*-iteration of* x_0 *under* g *converges to some* $\xi\in X$ *satisfying* $f\xi=p$.

Theorem 3.4 unifies the main fixed point results on weak contractions and on weak expansions in [3]. In fact, by putting $f=1_X$, we obtain

COROLLARY 1 [3, Theorem (1.4)] *Let* (X,d) *be complete and* $g : X \to X$ *a weak contraction, that is,* g *satisfies* (DG). *Then* g *has a fixed point* p, *and* $g^n x \to p$ *for each* $x\in X$.

By putting $g=1_X$ in Theorem 3.4, we have

COROLLARY 2 [3, Remark in Section 5] *Let* (X,d) *be complete and* $f : X \to X$ *be a surjective weak expansion, that is, there exists a compactly positive function* $\phi : R_+ \to R_+$ *such that for any* $x, y \in X$,

$$d(fx,fy) \geq d(x, y) + \phi(d(fx,fy)).$$

Then f *has a fixed point* p, *and,* $f^n x \to p$ *for each* $x\in X$.

4. Domain invariance for the Meir-Keeler type contractive fields

Let E be a Banach space and $U \subset E$ open. Given $F : U \to E$, the map $f : U \to E$ given by $fx = x - Fx$ is called the field (of displacements) associated with F.

LEMMA. *Let $F : E \to E$ be a selfmap of a metric space (E, d) satisfying* (Bd). *For any $r > 0$, if $d(x, Fx) \leq \delta(r)$ for some $x \in E$, then F maps $B(x, r+\delta(r))$ into itself.*

Proof. Suppose $y \in B(x, r+\delta(r))$. If $d(x, y) < r$, then $d(Fx, Fy) < r$, since F is contractive. If $r \leq d(x, y) < r+\delta(r)$, then $d(Fx, Fy) < r$, since F satisfies (Bd). Therefore, in any case,

$$d(x, Fy) \leq d(x, Fx) + d(Fx, Fy) < \delta(r) + r.$$

REMARK. Actually F maps $\overline{B}(x, r+\delta(r))$ into itself since F is continuous.

THEOREM 4.1. *Let E be a Banach space, $U \subset E$ open, $F : U \to E$ satisfy* (Bd), *and $f : U \to E$ its associated field. Then*

(a) *$f : U \to E$ is an open map, and*
(b) *$f : U \to fU$ is a homeomorphism.*

Proof. (a) For each $x \in U$ and a ball $B(x, r) \subset U$, we show that there is a ball $B(fx, \rho) \subset f(B(x, r))$. Suppose $0 < r' < r$. Then there exists $\delta(r') > 0$ satisfying (Bd) with respect to F.

Case (i) $r' + \delta(r') < r$: Choose any $x_0 \in B(fx, \delta(r'))$. Define $G : \overline{B}(x, r') \to E$ by $Gy = x_0 + Fy$. Then G also satisfies (Bd). Since

$$\|Gx - x\| = \|x_0 + Fx - x\| = \|x_0 - fx\| < \delta(r')$$

by Lemma, G maps $\overline{B}(x, r'+\delta(r'))$ into itself. Therefore, G has a unique fixed point y_0 in $\overline{B}(x, r'+\delta(r'))$. Since $y_0 = Gy_0 = x_0 + Fy_0$, we have $fy_0 = x_0$. Since x_0 is arbitrary in $B(fx, \delta(r'))$, we have

$$B(fx, \delta(r')) \subset f(\overline{B}(x, r'+\delta(r')) \subset f(B(x, r)).$$

Therefore f is open.

Case (ii) $r \leq r' + \delta(r')$: Let $\delta = (r - r')/2 > 0$. If $r' \leq \|x - y\| < r' + \delta$, then $\|Fx - Fy\| < r'$. As in the Case (i), we choose $x_0 \in B(fx, \delta)$ and define $G : \overline{B}(x, r') \to E$ by $Gy = x_0 + Fy$. Then G satisfies (Bd) and

$$\|Gx - x\| = \|x_0 + Fx - x\| = \|x_0 - fx\| < \delta.$$

Hence, by Lemma, G maps $\overline{B}(x, r'+\delta)$ into itself. Therefore, we have

$$B(fx, \delta) \subset f(\overline{B}(x, r'+\delta)) \subset f(B(x, r)),$$

and f is open.

(b) Let $fx = fy$. Then $x - Fx = y - Fy$ implies $x - y = Fx - Fy$. Since F is contractive, we should have $x = y$. Therefore f is injective. Since f is a continuous open bijection between U and fU, f is a homeomorphism.

REMARK. Note that Dugundji and Granas [3, Theorem 2. 1] obtained Theorem 4. 1 with respect to the condition (DG) instead of (Bd).

Using the above theorem, we can obtain some corollaries corresponding to results of Dugundji–Granas [3].

COROLLARY 1. *Let E be a Banach space and $F : E \to E$ satisfy* (Bd). *Then the associated field f is a homeomorphism of E onto itself.*

COROLLARY 2. *Let X be any space, E a Banach space and $f : X \to E$ an embedding of X onto an open set $U \subset E$. Let $g : X \to E$ be a map such that $g \circ f^{-1} : U \to E$ satisfies* (Bd). *Then $x \to fx - gx$ is also an open embedding of X into E.*

COROLLARY 3. [3, Proposition 2. 4] *Let E be a Banach space, $U \subset E$ open, and $f : U \to E$ a C^1–map. If its derivative $Df(x_0) : E \to E$ is an isomorphism, then f maps a neighborhood of x_0 homeomorphically onto a neighborhood of fx_0.*

Let (E, d) be a metric space. By imitating the definition of a contractive type map, we define the following expansive type map.

DEFINITION. Let (E, d) be a complete metric space. A map $f : E \to E$ is called a (Bd)–type expansive map if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq d(fx, fy) < \varepsilon + \delta(\varepsilon) \text{ implies } d(x, y) < \varepsilon.$$

Of course, such a map need not be continuous. If f is a surjective (Bd)–type expansive map, then f^{-1} is well defined and satisfies (Bd). Therefore we obtain a result by using Theorem 4. 1.

THEOREM 4. 2. *Let E be a Banach space and $F : E \to E$ a* (Bd)*–type expansive surjective map. Then the associated field f is a bijective open map.*

References

1. D.W. Boyd and J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464.
2. F.E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27-35.
3. J. Dugundji and A. Granas, *Weakly contractive maps and elementary domain invariance theorem*, Bull. Greek Math. Soc. **19** (1978), 141-151.
4. M. Hegedüs and T. Szilagyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japonica **25** (1980), 147-157.
5. M. Krasnoselskij [and others], *Approximate solution of operator equations*, Groningen, Wolters-Noordhoff.
6. A. Meir and E. Keeler, *A theorem on contraction mappins*, J. Math. Anal. Appl. **2** (1969), 526-529.
7. S. Park, *On general contractive type conditions*, J. Korean Math. Soc. **17** (1980), 131-140.
8. S. Park and J. S. Bae, *Extensions of a fixed point theorem of Meir and Keeler*, Arkiv fõr Mat. **19** (1981), 223-228.
9. S. Park and B. E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japonica **26** (1981), 13-20.
10. S. Park and K. P. Moon, *On generalized Meir-Keeler type contractive conditions*, J. Nat. Acad. Sci., Korea, Nat. Sci. Series **22**(1983), 31-41.

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