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REMARKS ON SUBSEQUENTIAL LIMIT POINTS OF A SEQUENCE OF ITERATES

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In this paper, we obtain results on the structure and property of the set of subsequential limit points of a sequence of iterates of a selfmap. In [3], Diaz and Metcalf initiated such study and, recently, Maiti and Babu [8] obtained a similar result on maps which are contractive over two consecutive elements of an orbit.

Our first result is a slightly extended version of [3, Theorem 6] and includes [3, Theorem 2]. This is used to obtain an extended form of the main result of [8]. Several remarks on our results are added.

Let f be a selfmap of a metric space (X, d) and $F(f)$ denote the set of fixed points of f. Further, for $x \in X$, $O(x)$ denotes the set of the sequence of iterates $\{f^n x\}_{n=0}^{\infty}$, where $f^0 x = x$, $\bar{O}(x)$ its closure, and $L(x)$ the set of subsequential limit points (or cluster points) of $\{f^n x\}$.

THEOREM 1. *Let f be a continuous selfmap of a metric space (X, d) and $x \in X$. Suppose that*

(i) *$\bar{O}(x)$ is compact, and*

(ii) *f is asymptotically regular at x, that is,*

$$\lim_n d(f^n x, f^{n+1} x) = 0.$$

Then $L(x)$ is a nonempty, closed and connected subset of $F(f)$, and either

(1) *$L(x)$ is a singleton, and $\lim_n f^n x$ exists and belongs to $F(f)$, or*

(2) *$L(x)$ is uncountable, and $L(x) \subset$ Bdry $F(f)$.*

Proof. The condition (i) ensures $L(x) \neq \phi$. It is easy to see that $p = fp$ for any $p \in L(x)$ since f is continuous and asymptotically regular at x (see [6, Theorem 1] or [5, Proposition 1]). Hence, $L(x) \subset F(f)$. Note that $L(x)$ is a closed subset of $F(f)$, which, because of the continuity of f, is also closed. The conditions (i) and (ii) ensure the connectedness of $L(x)$ as in [8, pp. 379-380] or [3, p. 484]. Since $L(x)$ is nonempty, closed, connected, and compact, it is either a singleton or uncountable (see [2, p. 96] or [3, pp. 467-468]). We claim that $d(f^m x, L(x)) \to 0$ as $m \to \infty$.

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Otherwise, there exist an $\varepsilon>0$ and a sequence $\{m_i\}_{i=1}^{\infty}$ of integers such that $d(f^{m_i}x,\ L(x))\geq\varepsilon$. Since $\bar{O}(x)$ is compact, $\{f^{m_i}x\}$ has a subsequence, say $\{f^{n_i}x\}_{i=1}^{\infty}$, converging to some $p\in L(x)$, whence we have $d(f^{n_i}x, p)<\varepsilon$ for sufficiently large i, a contradiction. This shows that if $L(x)$ is a singleton, then $\lim_m f^m x$ exists and belongs to $F(f)$. In case that $L(x)$ is uncountable, we have $L(x)\subset \text{Bdry } F(f)$ by the argument in [3, p. 469]. This completes our proof.

In Theorem 1, the condition (i) can be replaced by the compactness of fX, without affecting the conclusion. This result was obtained by Diaz and Metcalf [3, Theorem 6].

Note also that the main result of Diaz and Metcalf [3, Theorem 2] also follows from Theorem 1. For if $L(x)\neq\phi$, their assumptions imply that $\bar{O}(x)$ is compact (see [8, p. 379]) and that $\lim_n d(f^n x,\ f^{n+1}x)=0$ (see [3, p. 466]) for each $x\in X$.

Many variations and corollaries of Theorem 1 can be obtained as those of [3, Theorem 2], *e.g.*, in Theorem 1, if X is a nonempty closed subset of the real line, then only the case (1) holds.

In Theorem 1, if $O(x)$ is compact instead of (i), then (ii) implies the connectedness of $L(x)$, by a result of Barone [1]. This fact was generalized by Niechajewicz [9].

Theorem 1 can be regarded as a prototype of numerous extensions of the Banach contraction principle. The techniques used to obtain such extensions have been standard since Banach: place contractive conditions on maps so that suitable iterations (orbits) give Cauchy sequences, so well as one of continuity of the maps at the limit points, and another general fixed point (or coincidence) theorem results. Certain contractive type conditions on maps have two roles: first, they assure that certain iterations are Cauchy; and second, they assure the uniqueness of fixed point. In other words, such conditions assure (i) and (ii) in Theorem 1, and then assure the conclusion (1).

Further, it is well-known that if f is a continuous densifying selfmap of a metric space X and $O(x)$ is bounded for some $x\in X$, then $\bar{O}(x)$ is compact. Therefore, Theorem 1 is also a prototype of fixed point theorems on certain densifying maps, *e.g.*, see results in [7].

From Theorem 1, we obtain the following

THEOREM 2. *Let f be a continuous selfmap of a metric space (X, d) and $x\in X$. Suppose that*

(i) $\bar{O}(x)$ *is compact, and*

(iii) f *satisfies*

$$d(fy,\ f^2y) < d(y,\ fy)$$

for each $y \subset \bar{O}(x)$, $y \neq fy$.

Then the conclusion of Theorem 1 *holds.*

Proof. In view of Theorem 1 it suffices to show that (iii) implies (ii). Setting $c_i = d(f^i x,\ f^{i+1}x)$, we have $c_{i+1} \leq c_i$. Therefore, $c_i \to r$ as $i \to \infty$, where $r = \inf\{c_i\}$. For any $p \in L(x)$, there exists a subsequence $\{f^{i_k}x\}$ converges to p. Since f is continuous,

$$f^{i_k+1}x = f(f^{i_k}x) \to fp$$

and

$$f^{i_k+2}x = f^2(f^{i_k}x) \to f^2p$$

as $k \to \infty$. Thus we have

$$r = \lim_k d(f^{i_k}x,\ f^{i_k+1}x) = d(p,\ fp),$$
$$r = \lim_k d(f^{i_k+1}x,\ f^{i_k+2}x) = d(fp,\ f^2p).$$

Suppose $p \neq fp$. Then we have

$$d(fp,\ f^2p) < d(p,\ fp)$$

by (iii), which is impossible. Hence, we have $p = fp$, and (ii) holds. This completes our proof.

Theorem 2 is a particular case of [4, Theorem 1]. Note that the main result of Maiti and Babu [8, Theorem 3] follows from Theorem 2. Note also that a number of generalizations of the Banach contraction principle may follow from Theorem 2.

Therefore, Theorem 1 unifies and extends all the main results of [3] and [8].

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