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ON KASAHARA'S EXTENSION OF THE CARISTI-KIRK FIXED POINT THEOREM

Dedicated to the memory of Shouro Kasahara

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Abstract. Kasahara's extension of the Caristi-Kirk fixed point theorem with respect to a family of selfmaps of an L-space is generalized. Our generalization includes the Downing-Kirk fixed point theorem.

In [6], Kasahara generalized the Caristi-Kirk fixed point theorem [2] to a family of selfmaps of an L-space. On the other hand, Downing and Kirk [4] generalized the Caristi-Kirk theorem as follows:

Theorem A. (Downing-Kirk [4]) *Let X and Y be complete metric spaces and $g: X \to X$ an arbitrary map. Suppose there exist a closed map $f: X \to Y$, a lower semicontinuous function $\phi: fX \to R_+ = [0, \infty)$, and a constant $c > 0$ such that for each $x \in X$,*

$$\max \{d(x, gx), c\, d(fx, fgx)\} \leqslant \phi(fx) - \phi(fgx).$$

Then g has a fixed point.

If $X = Y$ and $f = 1_X$, the identity map of X, then Theorem A reduces to the Caristi-Kirk theorem. A number of useful applications of those results have appeared. For literature, see Caristi [3] and Park [7].

In this paper, we extend Theorem A following Kasahara's method. Consequently, we obtain a common extension of Kasahara's and Downing-Kirk's generalizations of the Caristi-Kirk theorem.

The following is basic:

Theorem B. (Kasahara [6]) *Let F be a family of selfmaps of a poset $(X, \leqslant)$ such that $gx \leqslant x$ for all $g \in F$ and for all $x \in X$. If each chain in X which contains*

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a fixed element $e \in X$ *has a lower bound, then* F *has a common fixed point.*

Definitions. A *Kasahara space* X is a nonempty L-space $(X, \to)$ which is d_X-complete for a premetric d_X on X such that the function $x \mapsto d_X(x, y)$ is lower semicontinuous for every $y \in X$ and $d_X(x, y) = 0$ implies $x = y$. We simply denote d instead of d_X.

Studies on such type of spaces were initiated by Kasahara [5], [6] relevant to fixed point theory.

A map $f: X \to Y$ between Kasahara spaces is said to be *closed* if, for any sequence $\{x_n\}_{n \in N}$ in X, the conditions $x_n \to x$ and $fx_n \to y$ imply $fx = y$.

Our main result is the following

Theorem 1. *Let* X *and* Y *be Kasahara spaces, and* $f: X \to Y$ *be a closed map. Let* F *be a family of selfmaps* g *of* X *and* $\phi: fX \to R_+$ *be a lower semicontinuous function such that for each* $g \in F$ *and for each* $x \in X$,

$$\max \{d(gx, x), d(fgx, fx)\} \leqslant \phi(fx) - \phi(fgx).$$

Then F *has a common fixed point.*

Proof. Define a relation $\leqslant$ in X by $x \leqslant y$ if and only if

$$\max \{d(x, y), d(fx, fy)\} \leqslant \phi(fy) - \phi(fx).$$

Then $(X, \leqslant)$ is a poset and that $gx \leqslant x$ for each $g \in F$ and for each $x \in X$. Let e be an arbitrary but given element of X, and C a chain in $(X, \leqslant)$ containing e. Let $\alpha = \operatorname{glb} \{\phi(fx) \mid x \in C\}$. If $\phi(fa) = \alpha$ for some $a \in C$, then a is a lower bound of C. For if $x \leqslant a$ for some x in $C \setminus \{a\}$, then we have

$$\max \{d(x, a), d(fx, fa)\} \leqslant \phi(fa) - \phi(fx) \leqslant 0,$$

which implies $d(x, a) = 0$, a contradiction. Now we consider the case when $\phi(fx) \neq \alpha$ for all $x \in C$. For each $n \in N$ and each $x \in C$, let $S(n, x)$ be the set of all $y \in C$ with $y \leqslant x$ and

$$\alpha < \phi(fy) < \alpha + \frac{1}{n}. \qquad (*)$$

Then each $S(n, x)$ is nonempty. In fact, there is a $y \in C$ satisfying $(*)$. If $y \leqslant x$, then $y \in S(n, x)$. If $x \leqslant y$, then

$$0 \leqslant \max \{d(x, y), d(fx, fy)\} \leqslant \phi(fy) - \phi(fx)$$

and, hence, $\alpha < \phi(fx) \leqslant \phi(fy) < \alpha + n^{-1}$, which shows that $x \in S(n, x)$. Let ψ be a choice function for the family of all nonempty subsets of C. Then by the recursion theorem, there is a sequence $\{x_n\}_{n \in N}$ in C such that $x_1 = e$ and $x_{n+1} = \psi(S(n, x_n))$ for each $n \in N$. Since $x_{n+1} \leqslant x_n$ for all $n \in N$, we have $\sum_{n=1}^{m} \max\{d(x_{n+1}, x_n), d(fx_{n+1}, fx_n)\} < \phi(fe)$ for each $m \in N$, and hence $\sum_{n=1}^{m} \max\{d(x_{n+1}, x_n), d(fx_{n+1}, fx_n)\} < \infty$. Since X and Y are d-complete, $x_n \to a$ for some $a \in X$ and $fx_n \to b$ for some $b \in Y$. Since f is closed, $fa = b$. Now let x be in C. Then we can find an $m \in N$ such that $\phi(fx_m) < \alpha + m^{-1} < \phi(fx)$. Since x and x_m are in the chain C, we know $x_m \leqslant x$. Therefore,

$$\max\{d(x_n, x), d(fx_n, fx)\} + \phi(fx_n) \leqslant \phi(fx) \qquad (**)$$

for every $n \geqslant m$. Since $d(\cdot, x)$, $d(\cdot, fx)$, and ϕ are lower semicontinuous on X, Y, and fX, respectively, it follows from $(**)$ that

$$\max\{d(a, x), d(fa, fx)\} + \phi(fa) \leqslant \phi(fx),$$

or equivalently $a \leqslant x$. This shows that a is a lower bound of C. In view of Theorem B, this completes our proof.

By putting $X = Y$ and $f = 1_X$ in Theorem 1, we obtain

Corollary 1. (Kasahara [6]) *Let X be a Kasahara space and F be a family of selfmaps of X. If there exists a lower semicontinuous function $\phi: X \to R_+$ such that for each $g \in F$ and for each $x \in X$,*

$$d(gx, x) \leqslant \phi(x) - \phi(gx)$$

then F has a common fixed point.

From Theorem 1, we obtain the following

Theorem 2. *Let X and Y be complete metric spaces, and $f: X \to Y$ be closed. Let F be a family of selfmaps g of X and $\phi: fX \to R_+$ a lower semicontinuous function such that for each $g \in F$ and for each $x \in X$*

$$\max\{d(gx, x), d(fgx, fx)\} \leqslant \phi(fx) - \phi(fgx).$$

Then F has a common fixed point.

The following result generalizes the Downing-Kirk theorem.

Corollary 2. *Let* X *and* Y *be complete metric spaces,* $f: X \to Y$ *be closed, and* $c > 0$. *Let* F *be a family of selfmaps* g *of* X *and* $\phi: fX \to R_+$ *a lower semicontinuous function such that for each* $g \in F$ *and for each* $x \in X$

$$\max \{d(gx, x), cd(fgx, fx)\} \leqslant \phi(fx) - \phi(fgx).$$

Then F *has a common fixed point.*

Proof. Let $k = \max \{1, c^{-1}\}$. Then

$$\max \{d(gx, x), d(fgx, fx)\} \leqslant k\phi(fx) - k\phi(fgx)$$

holds. Since $k\phi$ is lower semicontinuous, Corollary 2 follows from Theorem 2.

Remarks. Theorem 2 and Corollary 2 are actually equivalent. Various proofs of the Caristi-Kirk theorem are available, e.g., see Brezis-Browder [1], Penot [9], and Siegel [10]. Those can be modified in order to get other proofs of Theorem 1. However, we followed Kasahara's method faithfully in this paper. For some different proofs, see Park [8].

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