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A Note on a Theorem by Maiti and Babu

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In a recent paper [2], M. Maiti and A. C. Babu obtained a generalization of a theorem of J. B. Diaz and F. T. Metcalf [1]. In the present paper, we show that those results are actually equivalent.

We recall those results:

Theorem 1 ([1] Theorem 2). *Let* A *be a continuous self-map of a metric space* (S,d). *Suppose*

(i) $F(A) = \{ x \in S \mid A(x) = x \}$ *is nonempty and compact,*

(ii) *for each* $x \in S$, *with* $x \notin F(A)$, *one has* $d(A(x), F(A)) < d(x, F(A))$.

Then, for $x \in S$, *the set* $L(x)$ *is a closed connected subset of* $F(A)$. *Either* $L(x)$ *is empty, or a singleton, or uncountable. In case* $L(x)$ *is a singleton, then* $\lim A^m(x)$ *exists and belongs to* $F(A)$. *In case* $L(x)$ *is uncountable, then it is contained in the boundary of* $F(A)$.

In Theorem 1, $L(x)$ denotes the set of subsequential limit points of the sequence of iterates $\{A^m(x)\}_{m=1}^{\infty}$.

Theorem 2 ([2] Theorem 1). *Let* f *and* g *be selfmaps of a metric space* (X,d) *such that* $g(X) \subset f(X)$. *Assume that*

(i)′ f *and* g *are continuous,* f *is open and one-to-one,*

(ii)′ $F(f,g) = \{x \mid fx = gx \}$ *is nonempty and compact,*

and

(iii)' *for all* y *in* $X-F(f,g)$, $d(gy,G) < d(fy,G)$, *where* $G = gF(f,g) = fF(f,g)$.

For $x = x_0$ *in* X , *let* $\{x_n\}$ *be the sequence defined recursively by the relation* $gx_{n-1} = fx_n$. *Then the set* $L(x)$ *of subsequential limit points of* $\{x_n\}$ *is a closed and connected subset of* $F(f,g)$. $L(x)$ *is either empty, or a singleton, or uncountable. In case* $L(x)$ *is uncountable it is contained in the boundary of* $F(f,g)$. *In case* $L(x)$ *is a singleton,* $\lim x_n$ *exists and belongs to* $F(f,g)$.

Note that, by putting $f = 1_X$, Theorem 2 implies Theorem 1.

We show that Maiti-Babu's Theorem 2 is a simple consequence of Diaz-Metcalf's Theorem 1.

Consider the continuous map $A = gf^{-1} : fX \to fX$. Then

$$F(A) = \{x \in fX \mid x = gf^{-1}(x)\} = \{fy \in fX \mid fy = gy\} = G ,$$

since f is one-to-one and $f^{-1}x = y$ is equivalent to $x = fy$. Therefore, (i) follows from (i)' and (ii)' . Consider a point $y \in fX - F(A)$. By putting $f^{-1}y = y_1$, we have

$$d(A(y),F(A)) = d(gf^{-1}y,G) = d(gy_1,G) < d(fy_1,G) = d(y,F(A)) ,$$

from (iii)', where $y_1 \in X - F(f,g)$. Therefore, (ii) holds. Now, for any sequence $\{x_n\}$ in X satisfying $gx_{n-1} = fx_n$, we have a sequence $\{y_n\}$ in fX satisfying $fx_n = y_n$. This implies $A(y_{n-1}) = y_n$. Since $f^{-1} : fX \to X$ is a homeomorphism, to $L(x)$, there corresponds the set $L(y)$ of subsequential limit points of $\{y_n\}$ in fX . Therefore, by applying Theorem 1, we obtain the conclusions of Theorem 2.

Finally, we remark that other two results in [2] also can be obtained from the corresponding single map versions.

References

[1] J. B. Diaz and F. T. Metcalf : On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135 (1969), 459-485.

[2] M. Maiti and A. C. Babu : A generalization of a theorem of Diaz and Metcalf, Math. Semi. Notes 10 (1982), 135-141.

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