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Remarks on F.E. Browder's Fixed Point Theorems of Contractive Type

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F.E. Browder의 可縮型不動點定理에 관하여

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Abstract

F.E. Browder [1] posed a fixed point theorem of great generality and complexity such that a large part of the literature on contractive type maps can be subsumed under an intuitive and simple mode of argument. In this paper, we present sharper forms of such fixed point theorems which show that Browder's results can be stated in more general setting and include much more detailed cases than his.

The Banach contraction principle has long been one of the most important tools in the study of nonlinear problems. Motivated by this fact, during the past two decades, there has grown an extensive literatured devoted to sharper forms of the principle. Browder [1] noted that this literature had reache a point of scholarstic complexity and unreadability that its usefulness was open to serious question, and posed a fixed point theorem of great generality and complexity such that a large part of the literature can be subsumed under an intuitive and simple mode of argument. Note that similar attempts to Browder's are also shown by Hegedüs and Szilágyi [2] and Park [4].

However, there are still so many fixed point results of contractive type which can not be covered by Browder's theorem. It is, from one point of view, mainly because that Browder's contractive gauge functions seem to be too strict, that there are noncontinous maps of contractive type, and that there are results on non-complete metric spaces.

In this paper, we present sharper forms of such fixed point theorems which show that Browder's results can be stated in more general setting and include much more detailed cases than his.

We begin with following in Hegedüs and Szilágyi [2].

LEMMA 1. *For a subset P of $\boldsymbol{R}^2_+$, where $\boldsymbol{R}_+$ is the set of nonnegative real numbers, the following are equivalent:*

(i) *For any $\varepsilon>0$ there exist a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that $0\leqslant x<\varepsilon+\delta$ and $(x, y)\in P$ imply $y\leqslant\varepsilon_0$.*

(i)′ *There exists a nondecreasing map ϕ: $\boldsymbol{R}_+\to\boldsymbol{R}_+$ which is continuous from the right such that $\phi(t)<t$ for all $t>0$ and $(x, y)\in P$ implies $y\leqslant\phi(x)$.*

LEMMA 2. *For a $P\subset\boldsymbol{R}^2_+$ such that $(x,y)\in P$ and $x=0$ imply $y=0$, the following are equivalent:*

(ii) *For any $\varepsilon>0$ there exists a $\delta>0$ and an ε_0 with $0<\varepsilon_0<\varepsilon$ such that $\varepsilon\leqslant x<\varepsilon+\delta$ and $(x, y)\in P$ imply $y\leqslant\varepsilon_0$.*

(ii)′ *There exists a map $\phi:\boldsymbol{R}_+\to\boldsymbol{R}_+$ which is upper-semicontinuous from the right on $\boldsymbol{R}_+-\{0\}$ such that $\phi(t)<t$ for all $t>0$ and $(x, y)\in P$ implies $y\leqslant\phi(x)$.*

Note that, for a P in Lemma 2, conditions in Lemma 1 imply those in Lemma 2, and not conversely. We give a more extended condition.

LEMMA 3. *For a $P\subset\boldsymbol{R}^2_+$ such that $(x,y)\in P$ and $x=0$ imply $y=0$, any of (i), (i)′, (ii), and (ii)′ implies the following, and not conversely.*

(iii) *For any $\varepsilon>0$ there exists a $\delta>0$ such that $\varepsilon\leqslant x<\varepsilon+\delta$ and $(x, y)\in P$ imply $y<\varepsilon$.*

The condition (iii) is due to Meir and Keeler [3], where an example showing (iii)$\not\Rightarrow$ (ii)′ is given.

We now extend the key results of Browder [1].

LEMMA 4. *Let $\{d_n\}$ be a nonincreasing sequence in $\boldsymbol{R}_+$ satisfying the condition:*

(iv) *for any $\varepsilon>0$, there exists a $\delta>0$ such that*

$$\varepsilon\leqslant d_n<\varepsilon+\delta \text{ implies } d_{n+1}<\varepsilon.$$

Then $d_n\to 0$ as $n\to\infty$.

Proof. Since $\{d_n\}$ is nonincreasing by assumption, it converges to some $\varepsilon\geqslant 0$, Suppose $\varepsilon>0$. Then there exists $\delta>0$ satisfying (iv). Choose $n\in\omega$ such that $\varepsilon\leqslant d_n<\varepsilon+\delta$. Then $d_{n+1}<\varepsilon$, a contradiction.

Lemma 4 extends Lemma 2 of [3] and Lemma 1 of [1].

Throughout the remainder of this paper, X denotes a metric space with a metric d and f: $X\to X$ is a map. For a subset A of X, $d(A)$ denotes its diameter, and $\bar{A}$ its closure. For $x\in X$, $O(x)$ denotes the orbit of x under f, that is, $O(x)=\bigcup_{n\geqq 0}\{f^n x\}$, and $O(x,y)=O(x)$