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A GENERAL PRINCIPLE OF FIXED POINT ITERATIONS ON COMPACT INTERVALS*

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During the past three decades, there have appeared a number of results on iteration schemes which converge to fixed points of subsets of Banach spaces. In this paper, we show that certain type of asymptotic regularity of a sequence implies its convergence, and obtain a general method of finding iteration schemes converging to a fixed point of a continuous selfmap of a compact interval. Our method unifies and extends known ones of Mann [9], Franks and Marzec [6], and Rhoades [13], [14], [15]. Finally, we indicate some new applications to closed convex subsets of the Hilbert cube $\boldsymbol{H}_0$.

We begin with general observations.

PROPOSITION 1. *Let f be a selfmap of a metric space X. Suppose there exists a sequence $\{x_n\}$ in X such that*

(i) *a subsequence $\{x_{n_i}\}$ converges to some $p\in X$, and*

(ii) $d(x_n, x_{n+1}) \to 0$ *as* $n\to\infty$.

Then $x_{n_i+1}\to fp$ *iff* $p=fp$.

Proof. Since (ii) implies $d(x_{n_i}, x_{n_i+1})\to 0$, if $x_{n_i}\to p$ and $x_{n_i+1}\to q$, we must have $p=q$. Conversely, $d(x_{n_i+1}, p)\leqslant d(x_{n_i+1}, x_{n_i})+d(x_{n_i}, p)$ shows that $x_{n_i+1}\to p=fp$.

COROLLARY 1.1. *Let f be a continuous selfmap of a metric space X. Suppose there exists a subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ in X and a point $p\in X$ such that $x_{n_i}\to p$ and $x_{n_i+1}\to fp$. Then either $d(x_n, x_{n+1})\to 0$ or $d(x_n, fx_n)\to 0$ implies $p=fp$.*

Proof. The first case follows clearly from Proposition 1. For the second, since f is continuous, we have $fx_{n_i}\to fp$. Then $x_{n_i+1}\to fp$ implies

$$d(x_{n_i}, x_{n_i+1})\leqslant d(x_{n_i}, fx_{n_i})+d(fx_{n_i}, x_{n_i+1})\to 0$$

by assumption. Now the conclusion follows from Proposition 1.

A selfmap f of a metric space X is said to be *asymptotically regular* at $x\in X$ if $d(f^{n+1}x, f^n x)\to 0$ as $n\to\infty$ [2].

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COROLLARY 1.2. (Edelstein–O'Brien [5]) *If f is a continuous selfmap of a metric space X and asymptotically regular at an $x\in X$, then any cluster point of $\{f^n x\}$ is a fixed point of f.*

COROLLARY 1.3. (Belluce–Kirk [1]) *If f is a continuous compact selfmap of a metric space X and asymptotically regular at an $x\in X$, then f has a fixed point.*

In Corollaries 1.2 and 1.3, $\{f^n x\}$ does not necessarily converge to a fixed point of f.

PROPOSITION 2. *Let f be a selfmap of a metric space X. Suppose there exists a sequence $\{x_n\}$ in X such that*

(i) *a subsequence $\{x_{n_i}\}$ converges to some $p\in X$, and*

(ii) *$d(x_{n+1}, p)\leqslant d(x_n, p)$ for all n.*

Then $x_{n_i+1}\to fp$ iff $x_n\to p$ and $p=fp$.

Proof. For any n, there exists an n_i such that $n_i\leqslant n<n_{i+1}$. By (ii), we have $d(x_{n_{i+1}}, p)\leqslant d(x_n, p)\leqslant d(x_{n_i}, p)$, which implies $x_n\to p$ and $d(x_{n+1}, x_n)\to 0$. Now $x_{n_i+1}\to fp$ iff $p=fp$ by Proposition 1.

A selfmap f of a metric space X is said to be *quasi-nonexpansive* provided that $p\in X$ and $p=fp$ implies $d(fx, p)\leqslant d(x, p)$ for any $x\in X$ [3].

COROLLARY 2.1. *Let f be a continuous quasi-nonexpansive selfmap of a metric space X. Suppose there exists an $x\in X$ such that $\{f^n x\}$ has a cluster point $p\in X$. Then f is asymptotically regular at x iff $\{f^n x\}$ converges to p and $p=fp$.*

Hillam [7] showed that if X is a compact interval I, the quasi-nonexpansiveness in Corollary 2.1 can be removed. See Corollary 3.2 below.

COROLLARY 2.2. *Let f be a continuous quasi-nonexpansive selfmap of I. Then for any $t\in(0,1)$ and $x\in I$, $\{f_t^n x\}$ converges to a fixed point of f, where $f_t x=(1-t)x+t(fx)$.*

Proof. It is well-known that $f_t: I\to I$ is asymptotically regular (see Petryshyn–Williamson [11], or more generally, Dotson [4]).

For a compact interval $I=[a,b]$, Corollaries 2.1 and 1.3 can be strengthend as follows:

THEOREM 1. *Let f be a continuous selfmap of I, and $\{x_n\}$, $\{y_n\}$ sequences in I such that $y_n\in\overline{x_n(fx_n)}$ and $x_{n+1}\in\overline{x_n(fy_n)}$ for all $n\in\omega$.*

(1) *If $x_n-y_n\to 0$, then $x_n-x_{n+1}\to 0$ iff $\{x_n\}$ converges.*

(2) *$x_n-fx_n\to 0$ iff $\{x_n\}$ converges to a fixed point of f.*

In Theorem 1, $\overline{\quad}$ denotes the closed interval joining two points.

Proof. Clearly $x_n - x_{n+1} \to 0$ if $\{x_n\}$ converges. Conversely, assume $x_n - y_n \to 0$ and $x_n - x_{n+1} \to 0$. If $x_n = x_{n+1}$ except a finite number of n, then already $\{x_n\}$ converges; otherwise by eliminating such x_{n+1}'s we obtain a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \neq x_{n_{i+1}}$ for all i. Therefore, we may assume that $x_n \neq x_{n+1}$ if $x_n \neq fy_n$ and that $I = [0, 1]$. Now suppose that $\{x_n\}$ does not converge. Since I is compact, $\{x_n\}$ has at least two cluster points ξ_1 and ξ_2, $\xi_1 < \xi_2$. We claim that $fx = x$ for any $x \in (\xi_1, \xi_2)$. Pick any $x^* \in (\xi_1, \xi_2)$. If $fx^* > x^*$, then by the continuity of f there exists a δ in $(0, (x^* - \xi_1)/2)$ such that $fx > x$ for all x satisfying $|x - x^*| < \delta$, that is, $|x_n - x^*| < \delta$ implies $fx_n > x_n$. Since ξ_2 is a cluster point $\{x_n\}$, there exists an integer N such that $x_N > x^*$, $|x_n - y_n| < \delta/2$ and $|x_n - x_{n+1}| < \delta/2$ for all $n \geqslant N$. If $x_N \geqslant x^* + \delta/2$, then $x_{N+1} > x_N - \delta/2 \geqslant x^*$. If $x_N < x^* + \delta/2$, then $fx_N > x_N$, so that $fx_N \geqslant y_N \geqslant x_N > x^*$. Also $y_N < x_N + \delta/2$ so that $|y_N - x^*| < \delta$, and hence $fy_N > y_N$. This shows that $x_{N+1} \geqslant y_N \geqslant x_N$, that is, $x_{N+1} > x_N$ by assumption. Now by induction, each $x_n > x^*$ for $n \geqslant N$, contradicting that ξ_1 is a cluster point. In case $fx^* < x^*$, an analogous argument leads also a contradiction. Therefore $fx^* = x^*$ for all $x^* \in (\xi_1, \xi_2)$. Now note that $x_n \notin (\xi_1, \xi_2)$ for all n, otherwise $\{x_n\}$ converges since $x_n = fx_n = y_n = fy_n = x_{n+1}$ by hypothesis. Therefore, if $x_M \geqslant \xi_2$ for some integer M, then $x_n \geqslant \xi_2 > \xi_1$ for all $n \geqslant M$, and ξ_1 is not a cluster point. If $x_M \leqslant \xi_1$, ξ_2 is not a cluster point. This completes our proof of the first part. Now the second part follows from Proposition 1 and the first part.

Note that Theorem 1 also holds for a continuous bounded selfmap of the real line.

COROLLARY 3.1. *Let f be a continuous selfmap of I, and $\{x_n\}$ a sequence in I such that $x_{n+1} \in \overline{x_n(fx_n)}$* [resp. $x_{n+1} \in \overline{x_n(f^p x_n)}$ *for some given integer $p \geqslant 1$*] *for all $n \in \omega$. Then*

(1) *$x_n - x_{n+1} \to 0$ iff $\{x_n\}$ converges, and*

(2) *$x_n - fx_n \to 0$* [resp. $x_n - f^p x_n \to 0$] *iff $\{x_n\}$ converges to a fixed point of f* [resp. *to a periodic point of order p*].

Note that, in Corollary 3.1, $x_n - x_{n+1} \to 0$ does not imply that $\{x_n\}$ converges to a fixed point. For example, let $I = [0, 1]$, $fx = 1$ for all $x \in I$, and $x_n = 1/2 - 1/(n+1)$.

COROLLARY 3.2. (Hillam [7]) *Let f be a continuous selfmap of I. Then f is asymptotically regular at $x \in X$ iff $\{f^n x\}$ converges to a fixed point of f.*

In view of Theorem 1(2) or Corollary 3.1(2), we have a general method of finding a fixed point of a continuous selfmap f of I. For example, let us begin with any $x_0 \in I$. If $x_0 = fx_0$, we are done. Choose a point x_1 in $\overline{x_0(fx_0)}$ such that $x_1 \neq x_0$. Continuing this process, after choosing x_n, choose a point $x_{n+1} \in \overline{x_n(fx_n)}$ such that $x_{n+1} \neq fx_n$. If we could choose $\{x_n\}$ such that $x_n - fx_n \to 0$, then $\{x_n\}$ converges to a fixed point of f. One of the general methods of finding such sequence is known as the Mann iterative process using regular infinite matrices [9]. Such method will be useful to locate zeros of an eqation (e.g., Newton's method) or coincidence points of two functions by using computer with suitable programming.

We list some known results which are consequences of our method. Some of them also give elementary and constructive proofs of the Brouwer fixed point theorem for 1-dimensional case.

(I) (Mann [9] and Franks-Marzec [6]) Let f be a continuous selfmap of I. Then the sequence $\{x_n\}$ in $I = [a, b]$ given by the following iteration scheme converges to a fixed point of f:

$$\bar{x}_0 = x_0 \in I, \quad \bar{x}_{n+1} = fx_n, \quad x_n = \sum_{k=1}^{n} \bar{x}_k / n.$$

For, $x_{n+1} - x_n = (fx_n - x_n)/(n+1) \leqslant (b-a)/(n+1) \to 0$ and by using the regularity of the Cesàro matrix, we can show that $x_n - fx_n \to 0$ [9]. Now use Corollary 3.1.

(II) (Rhoades [13]) Given a continuous selfmap of I, the sequence $\{x_n\}$ defined by the following iteration scheme converges to a fixed point of f:

$$\bar{x}_0 = x_0 \in I, \quad \bar{x}_{n+1} = fx_n, \quad x_n = \sum_{k=1}^{n} a_{nk} \bar{x}_k,$$

where $a_{n0} > 0$, $a_{nk} \geqslant 0$ for $k > 0$, $\sum_{k=0}^{n} a_{nk} = 1$, $\sum_{n=1}^{\infty} a_{nn} = \infty$, and $a_{nn} \to 0$ as $n \to \infty$.

Note that $x_{n+1} \in \overline{x_n(fx_n)}$ and that

$$x_{n+1} - x_n = a_{n+1,n+1}(fx_n - x_n) \leqslant a_{n+1,n+1}(b-a) \to 0$$

implies $x_n \to p \in I$ by Corollary 3.1(1). However, the matrix $A = (a_{nk})$ is regular, hence $x_n - fx_n \to 0$ [13].

Similar iteration schemes have been defined by Reinermann [12], Outlaw and Groetsch [10], and Dotson [4].

(III) (Rhoades [14]) Given a continuous nondecreasing selfmap f of I, the sequence $\{x_n\}$ obtained by the following iteration scheme converges to a fixed point of f:

$$x_0 \in I, \qquad x_{n+1} = (1 - c_n) x_n + c_n f x_n, \qquad n \in \omega,$$

where $c_0 = 1$, $0 \leqslant c_n \leqslant 1$, and $\sum_{n=1}^{\infty} c_n = \infty$.

Note that $x_{n+1} \in \overline{x_n(fx_n)}$. If $x_0 = fx_0$, we are done. If $x_0 < fx_0$, by induction we know that $\{x_n\}$ is nondecreasing, and if $x_0 > fx_0$, $\{x_n\}$ is nonincreasing. Therefore, $\{x_n\}$ converges. As in (II), the iteration scheme is regular, and Corollary 3.1 works.

(IV) (Rhoades [15]) Given a continuous selfmap f of I, the sequence $\{x_n\}$ given by the following iteration scheme converges to a fixed point of f:

$$x_0 \in I,\ y_n = \beta_n f x_n + (1-\beta_n) x_n,\ x_{n+1} = \alpha_n f y_n + (1-\alpha_n) x_n,$$

where α_n, $\beta_n \in [0, 1]$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$, and $\Sigma \alpha_n = \infty$.

Note that $y_n \in \overline{x_n(fx_n)}$, $x_{n+1} \in \overline{x_n(fy_n)}$, $x_n - y_n \to 0$ since $\beta_n \to 0$, and $x_n - x_{n+1} \to 0$ since $\alpha_n \to 0$. Therefore, $x_n \to p \in I$ by Theorem 1(1). In fact, we have $p = fp$ [15].

The above scheme is due to Ishikawa [8].

In general, Theorem 1 does not hold for higher dimensions. See an example in [13].

Finally, consider a continuous selfmap f of a closed convex subset X of the Hilbert cube $\boldsymbol{H}_0$ satisfying

(*) *there exists a sequence of continuous maps* $f_n : I \to I$ *such that*

$$f(x^1, x^2, \cdots) = (f_1 x^1,\ f_2 x^2, \cdots)$$

for all $x = (x^1, x^2, \cdots) \in \boldsymbol{H}_0$.

THEOREM 2. *For a continuous selfmap f of X satisfying the condition* (*), *the I in* Theorem 1 *can be replaced by X without affecting the conclusion.*

Proof. Let Pr_i be the i-th projection. The conditions $x_n - y_n \to 0$ and $x_n - x_{n+1} \to 0$ imply $x_n{}^j - y_n{}^j \to 0$ and $x_n{}^j - x^j{}_{n+1} \to 0$ for each $j \geq 1$. Since $y_n{}^j \in \overline{x_n{}^j (fx_n)^j} = \overline{x_n{}^j (f_j x_n{}^j)}$, $x_{n+1}{}^j \in \overline{x_n{}^j (f_j y_n{}^j)}$ and f_j can be considered as a continuous selfmap of the compact interval $Pr_j\ X$, $\{x_n{}^j\}$ converges to some $p^j \in Pr_j\ X$, by Theorem 1(1). Let P_j denote the projection of $\boldsymbol{H}_0$ onto a j-dimensional subspace given by

$$P_j(x^1, x^2, \cdots) = (x^1, x^2, \cdots, x^j, 0, 0, \cdots).$$

Let $p = (p^1, p^2, \cdots) \in \boldsymbol{H}_0$. Since $p = \lim_j P_j(p)$ and $\lim_n P_j(x_n) = P_j(p)$, we have

$$p = \lim_j P_j(p) = \lim_j \lim_n P_j(x_n) = \lim_n \lim_j P_j(x_n) = \lim_n x_n.$$

This completes the proof of the essential part.

In view of Theorem 2, for a map satisfying the condition (*), the I in Corollaries 3.1 and 3.2 can be replaced by X without affecting their conclusions.

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