J. Korean Math. Soc.
Vol. 16, No. 2, 1980

A UNIFIED APPROACH TO FIXED POINTS OF CONTRACTIVE MAPS

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1. Introduction

A number of authors have defined various contractive type selfmaps of metric spaces which are generalizations of the well-known Banach contraction. In [20], Rhoades compared those contractive conditions and combined many known fixed point theorems.

The techniques used there have been standard since Banach: place contractive conditions on maps so that suitable iterations (orbits) give Cauchy sequences; introduce a hypothesis of completeness in the range containing those sequences, so well as one of continuity of the maps at the limit points, and another general fixed point (or coincidence) theorem results. The contractive conditions on maps have two roles: first, they assure that certain iterations are Cauchy; and second, they assure the uniqueness of fixed point. However, for the first role it is sufficient to assume either that the maps are contractive over two consecutive elements of an orbit, so that the orbit is asymptotically regular; or that the maps are contractive over the closure of an orbit, so that its limiting orbital diameter is zero.

Recently, Pal and Maiti [19] established fixed point theorems for maps which are contractive over two consecutive elements of an orbit. However, we show that their results follow essentially from a theorem of Edelstein [6] and the Banach contraction principle. Motivated by this fact, we show that most of contractive conditions imply either the orbit is asymptotically regular or the limiting orbital diameter is zero, in which cases we have simple fixed point theorems containing many known results.

In Section 2, we show that if a contractive condition implies that given orbit is asymptotically regular, then a fixed point theorem *w.r.t.* the condition is a consequence of our version of Edelstein's theorem.

In Section 3, we show that if a condition implies that the limiting orbital diameter is zero, then a fixed point theorem *w.r.t.* the condition is a consequence of a single result.

Received June 8, 1979

In fact, our purpose is a unified approach to fixed point theorems which generalize the Banach contraction principle, and, in Section 4, we show that any fixed point theorem *w.r.t.* contractive type maps satisfying one of the contractive conditions (1)~(24) and (26)~(49) in the list of Rhoades [20] and others in [5], [7], [8], [10], [16], [19], [23], and [24] follows from the following basic principle:

Let f be a selfmap of a topological space and d a lower semicontinuous, nonnegative real valued function defined on $X\times X$ such that $d(x,y)=0$ implies $x=y$. If there exists $u\in X$ such that $\lim_i d(f^i u, f^{i+1}u)=0$ and if $\{f^i u\}$ has a convergent subsequence with a limit $\xi\in X$ on which f is orbitally continuous, then ξ is a fixed point of f.

2. For asymptotically regular orbits

Let (X,d) be a metric space and f a selfmap of X. For $u\in X$, the orbit $\{u, fu, f^2u, \cdots\}$ of u generated by f will be denoted by $O(u)$. The closure of O will be denoted by $\bar{O}$. We say that f is *orbitally continuous* at $\xi\in X$ if $\lim_k f^{i_k}u=\xi$ implies $\lim_k f^{i_k+1}u=f\xi$. The space X is said to be *f-orbitally complete* if every Cauchys equence contained in $O(u)$ converges in X, for all $u\in X$. Following Browder and Petryshin [3], for a given $u\in X$, we say that the orbit $O(u)$ is *asymptotically regular* if $\lim_i d(f^i u, f^{i+1}u)=0$.

The following is the main result in this section.

THEOREM 1. *Let f be a selfmap of a metric space (X,d). If*

(i) *there exists a point $u\in X$ such that the orbit $O(u)$ has a cluster point $\xi\in X$,*

(ii) *f is orbitally continuous at ξ and $f\xi$, and*

(iii) *f satisfies*

$$d(x,y)>d(fx,fy)$$

for all $x, y=fx\in\bar{O}(u)$, $x\neq y$,

then ξ is a fixed point of f.

Proof. Setting $c_i=d(f^i u, f^{i+1}u)$, we have $c_{i+1}\leqslant c_i$. Therefore, $\{c_i\}$ is monotone decreasing and bounded also. Then $c_i\to l$ as $i\to\infty$, where $l=\inf\{c_i\}$. Since a subsequence $\{f^{i_k}u\}$ converges to $\xi\in X$, we have

$$f^{i_k+1}u=f(f^{i_k}u)\to f\xi$$

and

$$f^{i_k+2}u=f^2(f^{i_k+1}u)\to f^2\xi$$

as $k\to\infty$, because is orbitally continuous at ξ and $f\xi$. Thus we have

$$l=\lim_k d(f^{i_k}u, f^{i_k+1}u)=d(\xi, f\xi),$$

$$l=\lim_k d(f^{i_k+1}u, f^{i_k+2}u)=d(f\xi, f^2\xi).$$

Suppose $\xi\neq f\xi$. Then we have

$$d(f\xi, f^2\xi) < d(\xi, f\xi),$$

which is impossible. Hence we have $\xi = f\xi$.

REMARK 1. For a map satisfying $d(x, y) > d(fx, fy)$ for all $x, y \in X$, $x \neq y$, the condition (i) is needed in order to ensure that every such f possesses a fixed point (Rhoades [20], Theorem 2).

REMARK 2. In Theorem 1, if the inequality in (iii) holds for all $x, y \in X$, $x \neq y$, then f has a unique fixed point. Hence, we obtain Edelstein's theorem on contractive maps [6]. Note also that if $\bar{O}(u)$ or X is compact, the condition (i) is not necessary.

REMARK 3. Note that, in Theorem 1, $O(u)$ is asymptotically regular. In Theorem 1, d need not be a metric. If we assume that X is a topological space, d is a lower semicontinuous, nonnegative real valued function defined on $X \times X$ such that $d(x, y) = 0$ implies $x = y$, and ξ is a limit of a subsequence of $O(u)$, then Theorem 1 still holds.

Now we list some consequences of Theorem 1.

(1.1) Instead of (iii) in Theorem 1, Pal and Maiti ([19], Theorem 2) considered the following condition:

(iii)′ f satisfies one of the following inequalities for all $x, y \in \bar{O}(u)$, $x \neq y$.

(a) $d(x, fx) + d(y, fy) < 2\, d(x, y)$,

(b) $d(x, fx) + d(y, fy) < \frac{2}{3} \{d(x, fy) + d(y, fx) + d(x, y)\}$,

(c) $d(x, fx) + d(y, fy) + d(fx, fy) < \frac{3}{2} \{d(x, fy) + d(y, fx)\}$,

(d) $d(fx, fy) < \max \{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$.

Note that (iii)′ implies (iii). Also note that Pal, Maiti and Achari ([17], Theorem 1) is a consequence of Theorem 1.

(1.2) Note that the inequality $d(x, y) > d(fx, fy)$ in (iii) can be replaced by any inequality in the contractive conditions (1)$\sim$(11) and (18)$\sim$(22) in the list of Rhoades [20] without affecting the conclusion of Theorem 1, since any of them implies

(22) $d(fx, fy) < \max \{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\}$ for $x \neq y$; and, for any $x, y = fx$, $x \neq y$, (22) is equivalent to

$$d(fx, fy) < \max \{d(x, y), d(x, fy)/2\},$$

which reduces to our inequality $d(x, y) > d(fx, fy)$.

(1.3) Note also that any inequality in (14), (15), (23) of Rhoades [20] also can be placed in (iii) instead of our inequality, since (14)$\Rightarrow$(15)$\Rightarrow$(23) and (23) implies $d(x,y)>d(fx,fy)$ for $y=fx$ (See [20], p. 269).

(1.4) The condition (iii) can be replaced by the contractive condition of Wong [24]:

Suppose that there exist functions α_i, i=1, 2, 3, 4, 5, of $(0,\infty)$ into $[0,\infty)$ such that

(a) each α_i is upper semicontinuous from the right;

(b) $\sum_{i=1}^{5}\alpha_i(t)<t$, $t>0$;

(c) for any distinct x, y in $\bar{O}(u)$,

$$d(fx,fy)\leqslant a_1d(x,y)+a_2d(x,fx)+a_3d(y,fy)+a_4d(x,fy)+a_5d(y,fx)$$

where $a_i=\alpha_i(d(x,y))/d(x,y)$.

For, this condition implies (iii) (See the proof of [24], Theorem 1).

(1.5) The contractive condition:

(a) given $\varepsilon>0$, there exists $\delta>0$ such that

$$\varepsilon\leqslant d(x,y)<\varepsilon+\delta \quad \text{implies} \quad d(fx,fy)<\varepsilon,$$

considered by Meir and Keeler [16] also can be placed instead of the inequality in (iii). It is shown in [25] that (a) is equivalent to the following

(b) There exists a selfmap w of $[0,\infty)$ into $[0,\infty)$ such that $w(s)>s$ for all $s>0$, w is lower semicontinuous from the right on $(0,\infty)$ and

$$w(d(fx,fy))\leqslant d(x,y), \quad x,y\in X.$$

(1.6) Theorem 1 also contains results of Ciric ([5], Theorem 3), Husain and Sehgal [8], Corollaries 1 and 2), and Taskovitz ([23], Theorem 2).

The following is a modification of Theorem 1 and extends the Banach contraction principle.

THEOREM 2. *Let f be a selfmap of a metric space (X,d). If there exists a point $u\in X$ and a $\lambda\in[0,1)$ such that $\bar{O}(u)$ is complete and*

$$(*) \qquad d(fx,fy)\leqslant\lambda\, d(x,y)$$

holds for any x, $y=fx$ in $O(u)$, then $\{f^iu\}$ converges to some $\xi\in X$, and

$$d(f^iu,\xi)\leqslant\frac{\lambda^i}{1-\lambda}d(u,fu) \quad \text{for } i\geqslant 1.$$

Further, if f is orbitally continuous at ξ or if (*) holds for any x, y in $\bar{O}(u)$, then ξ is fixed under f.

Proof. Since $d(f^iu, f^{i+1}u) \leqslant \lambda d(f^{i-1}u, f^iu)$, we have

$$d(f^iu, f^{i+1}u) \leq \lambda^i d(u, fu) \text{ for } i > 1.$$

For any $i, j \geqslant 1$, we have

$$\begin{aligned} d(f^iu, f^{i+j}u) &\leq d(f^iu, f^{i+1}u) + \cdots + d(f^{i+j-1}u, f^{i+j}u) \\ &\leq d(f^iu, f^{i+1}u) \cdot (1 + \lambda + \cdots + \lambda^{j-1}) \\ &= \frac{1-\lambda^j}{1-\lambda} d(f^iu, f^{i+1}u) \leq \frac{1}{1-\lambda} d(f^iu, f^{i+1}u) \\ &\leq \frac{\lambda^i}{1-\lambda} d(u, fu). \end{aligned}$$

This shows that $\{f^iu\}$ is Cauchy and converges to some $\xi \in X$. By letting $j \to \infty$ in the above inequality, we have

$$d(f^iu, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, fu) \text{ for } i \geqslant 1.$$

Suppose f is orbitally continuous at ξ. Then $f^iu \to \xi$ implies $f^{i+1}u \to f\xi$. This shows that $\xi = f\xi$. Suppose (*) holds for any $x, y \in \overline{O}(u)$. Then

$$d(f^{i+1}u, f\xi) \leq \lambda d(f^iu, \xi)$$

for any i. This implies $\xi = f\xi$.

We list consequences of Theorem 2.

(2.1) Instead of the inequality $d(fx, fy) \leq \lambda d(x, y)$ in Theorem 2, Pal and Maiti ([19], Theorem 1) assumed that, for any two elements $x, y \in \overline{O}(u)$, at least one of the following is true.

(i) $d(x, fx) + d(y, fy) \leq \alpha d(x, y)$, $1 \leq \alpha < 2$,

(ii) $d(x, fx) + d(y, fy) \leq \beta \{d(x, fy) + d(y, fx) + d(x, y)\}$, $\frac{1}{2} \leq \beta \leq \frac{2}{3}$

(iii) $d(x, fx) + d(y, fy) + d(fx, fy) \leq \gamma \{d(x, fy) + d(y, fx)\}$, $1 \leq \gamma < \frac{3}{2}$,

(iv) $d(fx, fy) \leq \delta \max \{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$, $0 < \delta < 1$.

Then as the authors showed in [19], by taking

$$\lambda = \max\left\{\alpha - 1, \frac{2\beta - 1}{1-\beta}, \frac{\gamma - 1}{2-\gamma}, \delta\right\} < 1,$$

the inequality

$$d(fx, fy) \leq \lambda d(x, y)$$

holds for all $x, y=fx$ in any case (i), (ii), (iii), and (iv), and f is orbitally continuous at $\xi=\lim_i f^i u$. This shows that Theorem 2 implies ([19], Theorem 1). As was pointed out in [21], note also that the above contractive condition of Pal and Maiti is independent of (24) in [20] (See [21], p. 42).

(2.2) The inequality $d(fx,fy)\leq\lambda d(x,y)$ can be replaced by any inequality in the contractive conditions (2), (4), (5), (7), (8), (9), (11), (12), (14), (15), (18), (19), (21) and (23) in [20] without affecting the conclusion of Theorem 2. For (23), see p. 270 of [20]. It can be also replaced by the contractive conditions of Wong [24].

(2.3) Theorem 2 also contains results of Ćirić ([5], Theorem 1), Fisher ([7], Theorems 1 and 2), Jaggi ([10], Theorem 1), Pal and Maiti ([18], Theorems 1 and 3), and Taskovitz ([23], Theorem 1).

So far we have shown that all fixed point theorems with respect to the contractive conditions (1)∼(11), (14), (15) and (19)∼(23) in the list of Rhoades [20] are consequences of Theorems 1 and 2. For other contractive conditions, appropriate modification may work. For the contractive conditions (26)∼(36), (39), (40) and (43)∼(48) in [20], we have the following type of theorems.

THEOREM 3. *Let f be a selfmap of a metric space (X,d). If for some $p>0$,*
(i) *there exists a point $u\in X$ such that the orbit $O(u)$ under f^p has a cluster point $\xi\in X$, and*
(ii) *f satisfies*

$$d(x,y)>d(f^p x, f^p y) \tag{28}$$

for all $x,y\in X$, $x\neq y$,

then ξ is a unique fixed point of f.

Proof. Since f^p is continuous, by Theorem 1, f^p has a fixed point ξ, and it is clearly unique. Now

$$f\xi=f(f^p\xi)=f^p(f\xi)$$

shows that $f\xi$ is also a fixed point of f^p, whence we have $\xi=f\xi$. It is clear that f does not have another fixed point.

3. For orbits whose limiting orbital diameters are zero

Let f be a selfmap of a metric space (X,d). For each $u\in X$, let $O(f^n u)$ denote the sequence of iterates of $f^n u$, that is,

$$O(f^n u)=\bigcup_{i=n}^{\infty}\{f^i u\},\quad n=0,1,2,\cdots$$

where $f^0u=u$. In general, the sequence $\{\operatorname{diam} O(f^n u)\}$ is nonincreasing and has limit $r(u)\geqslant 0$. Following Belluce and Kirk [1], [2] we call the number $r(u)$ (which may be infinite) the *limiting orbital diameter* of f at u.

Now we have our main result in this section, which is motivated by [9]. Let $\boldsymbol{R}_+$ be the set of nonnegative real numbers.

THEOREM 4. *Let f be a selfmap of a metric space (X, d) satisfying the following conditions*:

(i) *there is a $u\in X$ such that $O(u)$ has a cluster point $\xi\in X$ and* $\operatorname{diam} O(u)$ $<\infty$.

(ii) *there is an upper semicontinuous map $\varphi:\boldsymbol{R}_+^5\to\boldsymbol{R}_+$ which is nondecreasing in each coordinate variable and satisfies the condition $\varphi(t,t,t,t,t)<t$ for any $t>0$ and the inequality*

$$d(fx,fy)\leq\varphi(d(x,y)d(x,fx),d(y,fy),d(x,fy),d(y,fx))$$

for all $x,y\in\overline{O}(u)$.

Then ξ is a fixed point of f and $f^iu\to\xi$.

Proof. By virtue of (i), $\delta_n=\operatorname{diam} O(f^n u)$ is finite for each n. Since $\delta_{n+1}\leq\delta_n$ for any n, $\{\delta_n\}$ converges to some $\delta\geqslant 0$. If $i,j\geqslant n+1$ then

$$\begin{aligned}d(f^iu,f^ju)&\leq\varphi(d(f^{i-1}u,f^{j-1}u),d(f^{i-1}u,f^iu),d(f^{j-1}u,f^ju),\\&\qquad d(f^{i-1}u,f^ju),d(f^{j-1}u,f^iu))\\&\leq\varphi(\delta_n,\delta_n,\delta_n,\delta_n,\delta_n),\end{aligned}$$

and hence we have $\delta_{n+1}\leq\varphi(\delta_n,\delta_n,\delta_n,\delta_n,\delta_n)$ for all n, which implies $\delta\leq\varphi(\delta,\delta,\delta,\delta,\delta)$ because of the upper semicontinuity of φ. So we have $\delta=0$, that is, the limiting orbital diameter at u is zero. Therefore, $\{f^iu\}$ is a Cauchy sequence, and hence $f^iu\to\xi$. Now

$$\begin{aligned}d(f^iu,f\xi)&\leq\varphi(d(f^{i-1}u,\xi),d(f^{i-1}u,f^iu);d(\xi,f\xi),\\&\qquad d(f^{i-1}u,f\xi),d(\xi,f^iu)\\&\leq\varphi(d(\xi,f\xi),d(\xi,f\xi),d(\xi,f\xi),d(\xi,f\xi)d,(\xi,f\xi))\end{aligned}$$

shows that $\xi=f\xi$.

REMARK. Note that $\xi=f\xi$ assures that $f|\overline{O}(u)$ is orbitally continuous at $\xi\in O(u)$. If the inequality in (ii) holds for all $x,y\in X$, then ξ is the unique fixed point of f, and for any $x\in X$, we have $f^ix\to\xi$.

For complete metric spaces we have the following consequence of Theorem 4.

THEOREM 5. *Let f be a selfmap of a metric space (X,d) satisfying the following conditions:*

(i) *there is a $u\in X$ such that $\bar{O}(u)$ is complete and* $\operatorname{diam} O(u)<\infty$.

(ii) *there is an upper semicontinuous map $\varphi:\boldsymbol{R}_+^5\to\boldsymbol{R}_+$ which is nondecreasing in each coordinate variable and satisfies the condition $\varphi(t,t,t,t,t)<t$ for any $t>0$ and the inequality*

$$d(fx,fy)\le\varphi(d(x,y),d(x,fx),d(y,fy),d(x,fy),d(y,fx))$$

for all $x,y\in\bar{O}(u)$.

Then f has a fixed point ξ and $f^iu\to\xi$.

REMARK. If X is f-orbitally complete and if the inequality in (ii) holds for all $x,y\in X$, then the fixed point is uinque and $f^ix\to\xi$ for all $x\in X$. Hence Theorem 5 extends a result of Husain and Sehgal [6].

We list some consequences of Theorem 5.

(5.1) If one defines

$$\varphi(x_1,x_2,x_3,x_4,x_5)=\lambda\max\{x_1,x_2,x_3,x_4,x_5\}$$

for some $\lambda\in[0,1)$, (ii) of Theorem 5 becomes the contractive condition (24) of Rhoades [20]. Therefore it can be replaced by any of the inequalities in the contractive conditions (1), (2), (4), (5), (7), (8), (9), (11), (12), (14), (15), (16), (18), (19), (21) and (23) in [20]. Note that the condition (24) was first considered by Ćirić [4] and Massa [15].

(5.2) Theorems 4 and 5 can obviously be extended to the corresponding contractive definitions involving some iterate of f. Therefore, instead of the condition (24), we may also use the conditions (26), (27), (29), (30), (32), (33), (34), (36), (38), (39), (40), (41), (43), (44), (46), (48), and (49) in [20].

4. A unified approach

In Section 2. we showed that Theorems 1 and 2 have many consequences with respect to contractive conditions which imply that the orbits are asymptotically regular. On the other hand, in Section 3. Thorem 4 extends theorems with respect to contractive conditions which imply that the limiting orbital diameters are zero. In this section, we show that the following implies any of Theorems 1, 2 and 4.

THEOREM 6. *Let f be a selfmap of a topological space X and d a lower semicontinuous, nonnegative real valued function defined on $X\times X$ such that $d(x,y)=0$ implies $x=y$. If there exists $u\in X$ such that $\lim_i d(f^iu,f^{i+1}u)=0$*

and if $\{f^i u\}$ *has a convergent subsequence with a limit* $\xi \in X$ *on which* f *is orbitally continuous, then* ξ *is a fixed point of* f.

Proof. Suppose a subsequence $\{f^{i_k}u\}$ of $\{f^i u\}$ converges to ξ. Since $\lim_i d(f^i u, f^{i+1}u)=0$ and for any sufficiently large i there exists i_k such that $i_k \geqslant i$, we have

$$0=\lim_i d(f^i u, f^{i+1}u)=\lim_k d(f^{i_k}u, f^{i_k+1}u)\geqslant d(\xi, f\xi)$$

because $f^{i_k}u \to \xi$, $f^{i_k+1}u \to f\xi$ from the orbital continuity of f at ξ, and d is lower semicontinuous. This shows that $d(\xi, f\xi)=0$.

In the proofs of Theorems 1 and 2, it was shown that $O(u)$ is asymptotically regular. Hence Theorems 1 and 2 follow from Theorem 6.

It is easy to see that any orbit whose limiting orbital diameter is zero is asymptotically regular. Hence we have the following

COROLLARY. *Let* f *be a selfmap of a topological space* X *and* d *a lower semicontinuous, nonnegative real valued function defined on* $X \times X$ *such that* $d(x, y)=0$ *implies* $x=y$. *If there exists* $u \in X$ *whose limiting orbital diameter under* f *is zero and if* ξ *is a limit of a subsequence of* $\{f^i u\}$, *on which* f *is orbitally continuous, then* ξ *is a fixed point of* f.

Here the meaning of the limiting orbital diameter is properly modified. Corollary contains Theorem 2.1 and 2.2 of Belluce and Kirk [2].

Now it remains to show that Theorem 4 follows from Corollary. In fact, in Theorem 4, the limiting orbital diameter at u is zero, and it was noted that $f|\bar{O}(u)$ is orbitally continuous at $\xi \in \bar{O}(u)$

Therefore, all of the main results of this paper are consequences of Theorem 6.

5. Conclusion

So far from Theorem 6 we have derived that if f is a selfmap of a complete metric space X satisfying one of the contractive conditions (1)∼(24), (26)∼(49) in [20] and some others in [5], [7], [8], [10], [16], [19], [23] and [24], then f has a (unique) fixed point ξ and $f^i x \to \xi$ for any $x \in X$. In certain cases, e.g. (24) and (49), there are also approximation formulas.

We now conclude this paper by indicating for further work.

(i) For the contractive conditions (50)∼(74), (75)∼(99) and (100)∼(124) in [20], similar arguments to this paper may be possible.

(ii) For pairs of maps satisfying the contractive conditions (126)∼(250) in [20], some basic principle like our Theorem 6 may work.

(iii) Some results in this paper can be rewritten for spaces more general than metric ones, e.g., L-spaces [12], generalized metric spaces [4], and Hausdorff uniform spaces [22], and 2-metric spaces.

(iv) Extending Jungck's generalization [11] of the Banach contraction principle, some results in this paper can be extended to common fixed point of commuting maps.

(v) Theorem 6 and its consequences may have some applications to Banach spaces.

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