ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Number 4, Fall 1978

FIXED POINTS OF f-CONTRACTIVE MAPS

SEHIE PARK

1. **Introduction.** Let (X, d) be a metric space. A fixed point of a map $g : X \to X$ is a common fixed point of g and the identity map 1_X of X. Motivated by this fact, we replace 1_X by a continuous map $f : X \to X$, and obtain the following.

DEFINITIONS. Let f be a continuous self-map of X. Then a self-map g of X is said to be *f-contractive* if $d(gx, gy) < d(fx, fy)$ for all $x, y \in X$, $gx \neq gy$.

Let C_f denote the family of all maps $g : X \to X$ such that $gX \subset fX$ and $gf = fg$. Given a point $x_0 \in X$ and a map $g \in C_f$, *an f-iteration of x_0 under g* is a sequence $\{fx_n\}_{n=1}^{\infty}$ such that $fx_n = gx_{n-1}$.

We observe that an f-contractive map is always continuous. Note that given $x_0 \in X$, its f-iteration under g is not unique; however, in case $f = 1_X$, these definitions reduce to the usual ones.

We give conditions under which f-contractive maps have fixed points. In fact, necessary and sufficient conditions for the existence of fixed points of continuous self-maps of X are given. In order to do this, criteria for an f-iteration to be Cauchy are of interest. In this direction, Geraghty [5] obtained important results on usual contractive maps and iterations.

In this paper, we generalize results of Edelstein [4], Rakotch [7], and Geraghty [5] on the existence of fixed points, and, consequently, obtain many extended forms of the Banach contraction principle, especially those of Boyd-Wong [2], [8], Geraghty [5], Jungck [6], and Rakotch [7].

In § 2, basic n.a.s.c.'s for the existence of fixed points of self-maps of an arbitrary metric space and their applications are given.

In § 3, we give a n.a.s.c. that an f-iteration of $x_0 \in X$ under g be convergent. This condition is used to prove criteria for the existence of fixed points for metric spaces more general than complete ones. Some applications are also considered.

Throughout this paper, X denotes a metric space with metric d, and f denotes always a continuous self-map of X.

2. **General existence theorems.** In this section, we give some n.a.s.c.'s for the existence of fixed points of a continuous self-map f of X. First, we need the following.

Received by the editors on November 16, 1976.

AMS(MOS) subject classifications (1970). Primary 54H25; Secondary 47H10.

LEMMA 2.1. *Let f and g be commuting self-maps of a metric space X. If g^N is f-contractive for some integer $N > 0$ and f, g^N have a point of coincidence $\zeta \in X$, then $f\zeta$ is the unique common fixed point of f and g.*

PROOF. Let $\eta = f\zeta = g^N\zeta$ and $\eta \neq f\eta$. Then $d(\eta, f\eta) = d(g^N\zeta, g^N f\zeta) < d(f\zeta, ff\zeta) = d(\eta, f\eta)$ leads to a contradiction. Therefore $\eta = f\eta = g^N\eta$. Suppose f and g^N have another common fixed point η'. Then $d(\eta, \eta') = d(g^N\eta, g^N\eta') < d(f\eta, f\eta') = d(\eta, \eta')$ leads to another contradiction. Therefore η is the unique common fixed point of f and g^N. But from $\eta = f\eta = g^N\eta$ we have $g\eta = f(g\eta) = g^N(g\eta)$, whence $g\eta = \eta$. Thus η is a common fixed point of f and g. Now, η is unique since $\eta' = f\eta' = g\eta'$ implies $\eta' = f\eta' = g^N\eta'$.

The following is basic in this paper.

THEOREM 2.2. *A continuous self-map f of X has a fixed point iff there exists an f-contractive map g in C_f such that for some $x_0 \in X$, an f-iteration $\{fx_n\}$ of x_0 under g has a subsequence $\{fx_{n_i}\}$ converging to a point $\zeta \in X$. Indeed, f and g have a unique common fixed point $f\zeta$.*

PROOF. Suppose that $f\eta = \eta$ for some $\eta \in X$. Define $g: X \to X$ by $gx = \eta$ for all $x \in X$. Then clearly $g \in C_f$ and g is f-contractive. Note that for any $x \in X$, its f-iteration under g converges to η and η is the unique common fixed point of f and g.

Conversely, from the continuities of f, g and $fx_{n_i} \to \zeta$, we have $ffx_{n_i} \to f\zeta$ and $gfx_{n_i} \to g\zeta$. We define a function $r: Y = fX \times fX - \Delta \to \mathbf{R}$ by $r(fp, fq) = d(gp, gq)/d(fp, fq)$, where Δ denotes the diagonal of X. Then r is continuous and $r(fp, fq) < 1$. Thus if $f\zeta \neq g\zeta$, there is an α, $0 < \alpha < 1$, and an open set U of Y such that $(f\zeta, g\zeta) \in U$ and if $(fp, fq) \in U$ then $0 \leqq r(fp, fq) < \alpha$. Now choose $\rho > 0$ so that (1) $\rho < (1/3)d(f\zeta, g\zeta)$ and (2) if $B_1 = B(f\zeta, \rho)$ and $B_2 = B(g\zeta, \rho)$ are open balls, then $B_1 \times B_2 \subset U$. Since $ffx_{n_i} \to f\zeta$ and $gfx_{n_i} \to g\zeta$, there exists $N > 0$ such that $i > N$ implies $ffx_{n_i} \in B_1$ and $gfx_{n_i} \in B_2$. Therefore $d(ffx_{n_i}, gfx_{n_i}) > \rho$ for all $i > N$. On the other hand, from the definition of r, the choice of U, and the fact that $ffx_{n_i} = gfx_{n_i - 1}$, we have

$$d(ffx_{n_i+1}, ffx_{n_i+2}) < \alpha\, d(ffx_{n_i}, ffx_{n_i+1}).$$

Further, if $\ell > j > N$, then

$$d(ffx_{n_\ell}, ffx_{n_\ell+1}) \leqq d(ffx_{n_{\ell-1}+1}, ffx_{n_{\ell-1}+2}) < \alpha d(ffx_{n_{\ell-1}}, ffx_{n_{\ell-1}+1}).$$

Then by repeating this argument we get

$$d(ffx_{n_\ell}, ffx_{n_\ell+1}) < \alpha^{\ell-j} d(ffx_{n_j}, ffx_{n_j+1}).$$

But with fixed j, $\alpha^{\ell-j} \to 0$ as $\ell \to \infty$, whence $d(ffx_{n_\ell}, ffx_{n_\ell+1}) \to 0$. This contradicts $d(ffx_{n_\ell}, gfx_{n_\ell}) > \rho$ for $\ell > N$. Thus we conclude that $f\zeta = g\zeta$, and, by Lemma 2.1, $\eta = f\zeta$ is the unique common fixed point of f and g.

If $f = 1_X$, Theorem 2.2 is reduced to a theorem of Edelstein [4].

From Lemma 2.1 and Theorem 2.2, we have

COROLLARY 2.3. *A continuous self-map f of a compact metric space X has a fixed point iff there exists a map g in C_f such that for some integer $N > 0$, g^N is f-contractive. Indeed, f and g have a unique common fixed point.*

In case $f = 1_X$, Corollary 2.3 is a particular case of a result of Bailey [1], and is reduced to a result of Edelstein [4] whenever $N = 1$.

THEOREM 2.4. *A continuous self-map f of a metric space X has a fixed point iff there exists an f-contractive map g in C_f, a subset $M \subset X$ and a point $x_0 \in M$ such that*

$$d(fx, fx_0) - d(gx, gx_0) \geqq 2d(fx_0, gx_0) \tag{1}$$

for every $x \in X - M$ and g maps M into a compact subset of X. Indeed, f and g have a unique common fixed point.

PROOF. Suppose that $f\eta = \eta$ for some $\eta \in X$. Define $g: X \to X$ by $gx = \eta$ for all $x \in X$. Then g is in C_f and f-contractive. Putting $x_0 = \eta$ and $M = \{\eta\}$, the necessity follows.

Conversely, if $fx_0 = gx_0$, it is the unique common fixed point of f and g, by Lemma 2.1. Suppose $fx_0 \neq gx_0$ and let $\{fx_n\}_{n=1}^{\infty}$ be an f-iteration of x_0 under g. Since g maps M into a compact set by assumption, by Theorem 2.2, it suffices to show that $x_n \in M$ for every n. Since g is f-contractive, if $fx_{n-1} = fx_n$, i.e., $gx_{n-1} = fx_{n-1}$, for some n, then, by Lemma 2.1, f and g already have a unique common fixed point. Hence we may assume that $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$ for all n. From $fx_0 \neq gx_0$, it follows that $d(fx_n, fx_{n+1}) < d(fx_0, gx_0)$ for all n. Then

$$d(fx_n, fx_0) \leqq d(fx_n, gx_n) + d(gx_n, gx_0) + d(gx_0, fx_0)$$

implies

$$d(fx_n, fx_0) - d(gx_n, gx_0) < 2d(fx_0, gx_0).$$

Thus, by (1), we have $x_n \in M$ for all n.

If $x_0 \notin M$, Theorem 2.4, then the existence of a common fixed point of f and g follows immediately by putting $x = x_0$ in (1). In case $f = 1_X$, Theorem 2.4 is due to Rakotch [7, Theorem 1].

Now, following Rakotch [7], we introduce a class of functions.

DEFINITION. $\mathscr{S}$ is the class of monotonically decreasing functions $\alpha : (0, \infty) \to [0, 1)$.

For $\alpha \in \mathscr{S}$, let $\alpha(x, y) = \alpha(d(fx, fy))$.

COROLLARY 2.5. *A continuous self-map f of X has a fixed point iff there exists an f-contractive map g in C_f and a point $x_0 \in X$ satisfying*

$$d(gx, gx_0) \leqq \alpha(x, x_0)d(fx, fx_0) \tag{2}$$

for every $x \in X$, $fx_0 \neq fx$, where $\alpha \in \mathscr{S}$ and g maps the open ball $B(fx_0, r)$ with $r = 2d(fx_0, gx_0)/[1 - \alpha(2d(fx_0, gx_0))]$ into a compact subset of X. Indeed, f and g have a unique common fixed point.

PROOF. Suppose that $f\eta = \eta$ for some $\eta \in X$. Defining $g : X \to X$ by $gx = \eta$ and putting $x_0 = \eta$, for some constant $\alpha \in (0, 1)$, everything is trivially satisfied.

Conversely, in Theorem 2.4, take $M = B(fx_0, r)$, then from (2), by the definition of $\alpha(d)$ and $r \geqq 2d(fx_0, gx_0)$, it follows that if $d(fx, fx_0) \geqq r$ then

$$\begin{aligned} d(fx, fx_0) - d(gx, gx_0) &\geqq d(fx, fx_0) - \alpha(x, x_0)d(fx, fx_0) \\ &= [1 - \alpha(x, x_0)]d(fx, fx_0) \\ &\geqq [1 - \alpha(r)]r \\ &> [1 - \alpha(2d(fx_0, gx_0))]r \\ &= 2d(fx_0, gx_0), \end{aligned}$$

that is, (1) holds.

In case $f = 1_X$, Corollary 2.5 is due to Rakotch [7].

3. g-orbitally complete spaces only.

Given a continuous self-map f of X, we introduce a condition on X somewhat more general than completeness.

DEFINITION. Given $g \in C_f$, X is said to be *g-orbitally complete w.r.t. f* if, for any $x \in X$, every Cauchy subsequence of an arbitrary f-iteration $\{fx_n\}_{n=1}^{\infty}$ of x under g converges in X.

The g-orbital completeness w.r.t. 1_X is just the g-orbital completeness of Ćirić [3].

For any pair of sequences $\{x_n\}$ and $\{y_n\}$ in X with $fx_n \neq fy_n$, we write

$$d_n = d(fx_n, fy_n) \text{ and } \Delta_n = d(gx_n, gy_n)/d_n.$$

We have the following theorem.

THEOREM 3.1. *Let f be a continuous self-map of X and g be an f-contractive map in C_f such that X is g-orbitally complete w.r.t. f. Let $x_0 \in X$ and $\{fx_n\}_{n=1}^{\infty}$ be an f-iteration of x_0 under g. Then $\{fx_n\}$ converges to a point $\zeta \in X$ and hence $f\zeta$ is the unique common fixed point of f and g iff, for any two subsequences $\{fx_{h_n}\}$ and $\{fx_{k_n}\}$ with $fx_{h_n} \neq fx_{k_n}$, we have that $\Delta_n \to 1$ implies $d_n \to 0$.*

PROOF. Suppose $fx_n \to \zeta$ and let $\{fx_{h_n}\}$ and $\{fx_{k_n}\}$ be any two subsequences. Then $d_n = d(fx_{h_n}, fx_{k_n}) \to 0$ and the condition is satisfied.

Conversely, assume the condition is satisfied for a given point $x_0 \in X$. Since g is f-contractive, if $fx_{n+1} = gx_n = fx_n$ for some n, then f and g already have a unique common fixed point by Lemma 2.1. Hence we may assume that $d(fx_n, fx_{n+1}) = d(gx_{n-1}, gx_n) < d(fx_{n-1}, fx_n)$ for all n. Now $d_n = d(fx_n, fx_{n+1})$ is a decreasing sequence of positive numbers and so approaches some $\epsilon \geqq 0$. Assume $\epsilon > 0$. Then letting $h_n = n$ and $k_n = n + 1$, we have $d_n \to \epsilon > 0$ while $\Delta_n \to 1$. This leads to a contradiction. Hence $d(fx_n, fx_{n+1}) \to 0$. Now assume that $\{fx_n\}$ is not Cauchy. Then there exists some $\epsilon > 0$ such that every tail $\{fx_n\}_{n \geqq N}$ has diameter $D_N = \sup_{n,m \geqq N} d(fx_n, fx_m) > \epsilon$. Given this ϵ, we will construct a pair of subsequences violating the condition. For any $n > 0$, let N_n be so large that $d(fx_m, fx_{m+1}) < 1/n$ for all $m \geqq N_n$, as is possible since $d(fx_m, fx_{m+1}) \to 0$. Let $h_n \geqq N_n$ be the smallest integer such that for some $k_n > h_n$, $d(fx_{h_n}, fx_{k_n}) > \epsilon$. Such pairs exist by the above diameter condition. Next choose k_n to be the smallest such integer $> h_n$. Then $d(fx_{h_n}, fx_{k_n - 1}) \leqq \epsilon$ and $\epsilon \leqq d_n = d(fx_{h_n}, fx_{k_n}) < \epsilon + 1/n$. Moreover, we have

$$1 \geqq \Delta_n = d(gx_{h_n}, gx_{k_n})/d_n \geqq (d_n - 2/n)/d_n.$$

So $\Delta_n \to 1$ while $d_n \to \epsilon > 0$, again leading to a contradiction. So $\{fx_n\}$ must be a Cauchy sequence and converges in X since X is g-orbitally complete w.r.t. f. Now by Theorem 2.2., our proof is complete.

The above proof is essentially that of Theorem 1.1 of Geraghty [5], which is a particular case $f = 1_X$. By thoroughly examining the proof, we also obtain the following extended form of Corollary 1.2 of [5].

COROLLARY 3.2. *Let f be a continuous self-map of X and g be an f-contractive map in C_f such that X is g-orbitally complete w.r.t. f. Let $x_0 \in X$ and $\{fx_n\}_{n=1}^{\infty}$ be an f-iteration of x_0 under g. Then $\{fx_n\}$ converges to a point $\zeta \in X$, and hence $f\zeta$ is the unique common fixed point of f and g iff, for any two subsequences $\{fx_{h_n}\}$ and $\{fx_{k_n}\}$ with $fx_{h_n} \neq fx_{k_n}$, we have that $\Delta_n \to 1$, with d_n decreasing, implies $d_n \to 0$.*

According to Geraghty [5], we can convert this sequential condition to the more customary functional form.

DEFINITION. $\mathscr{T}$ is the class of functions $\alpha : (0, \infty) \to [0, 1)$ such that
(i) $\alpha(t_n) \to 1$ implies $t_n \to 0$. [5]. As before, $\alpha(x, y) = \alpha(d(fx, fy))$ when $\alpha \in \mathscr{T}$.

REMARK. Note that $\mathscr{S} \subset \mathscr{T}$ and that we do not assume any continuity on α. Using Corollary 3.2, we can replace (i) by
(ii) $\alpha(t_n) \to 1$ with t_n decreasing implies $t_n \to 0$ [5].
Note also that any continuous $\alpha : (0, \infty) \to [0, 1)$ is contained in $\mathscr{T}$ (cf. Corollary 3.8).

THEOREM 3.3. *Let f be a continuous self-map of X and g be an f-contractive map in C_f such that X is g-orbitally complete w.r.t. f. Let $x_0 \in X$ and $\{fx_n\}_{n=1}^{\infty}$ be an f-iteration of x_0 under g. Then $\{fx_n\}$ converges to some $\zeta \in X$ and hence $f\zeta$ is the unique common fixed point of f and g iff there exists an α in $\mathscr{T}$ such that for all n, m, $fx_n \neq fx_m$, we have*

$$d(gx_n, gx_m) \leqq \alpha(x_n, x_m)d(fx_n, fx_m).$$

PROOF. It suffices to show that the existence of such an α in $\mathscr{T}$ is equivalent to the sequential condition of Theorem 3.1. Suppose such an α exists. Let $\{fx_{h_n}\}$ and $\{fx_{k_n}\}$ be subsequences with $fx_{h_n} \neq fx_{k_n}$. Assume that $\Delta_n \to 1$. Then it follows from the above inequality that $\alpha(x_{h_n}, x_{k_n}) \to 1$. But then since $\alpha \in \mathscr{T}$, we have $d(fx_{h_n}, fx_{k_n}) \to 0$.
Conversely, suppose that the sequential condition holds. Define $\alpha : (0, \infty) \to [0, \infty)$ as follows:

$$\alpha(t) = \sup \{d(gx_n, gx_m)/d(fx_n, fx_m) \mid d(fx_n, fx_m) \geqq t\}$$

if $d(fx_n, fx_m) \geqq t$ holds for some m, n; and $\alpha(t) = 0$ otherwise. Since g is f-contractive, the quotients are all < 1 and so α is defined for all $t > 0$ and $\alpha \leqq 1$. Now assume that $\alpha(t_n) \to 1$ for $t_n \in (0, \infty)$. We may further assume without loss of generality that $1 - 1/n < \alpha(t_n) \leqq 1$. Now we have to show $t_n \to 0$. By the definition of $\alpha(t_n)$, for each $n > 0$, there is a pair fx_{h_n}, fx_{k_n} in $\{fx_n\}$ with

$$d(fx_{h_n}, fx_{k_n}) \geqq t_n \text{ and}$$
$$1 - 1/n < d(gx_{h_n}, gx_{k_n})/d(fx_{h_n}, fx_{k_n}) \leqq \alpha(t_n).$$

So $\Delta_n \to 1$. But then by the sequential condition of Theorem 3.1, $d(fx_{h_n}, fx_{k_n}) \to 0$. So $t_n \to 0$. This completes our proof.

In case $f = 1_X$, Theorem 3.3 is reduced to Theorem 1.3 of Geraghty [5].

As consequences of Theorem 3.3 we obtain fixed point theorems.

THEOREM 3.4. *A continuous self-map f of a metric space X has a fixed point iff there exists an f-contractive map g in C_f such that X is g-orbitally complete w.r.t. f, and there exists a subset $M \subset X$ and a point $x_0 \in M$ satisfying the following:*

(1) *$d(fx, fx_0) - d(gx, gx_0) \geqq 2d(fx_0, gx_0)$ for every $x \in X - M$,*

(2) *$d(gx, gy) \leqq \alpha(x, y)d(fx, fy)$ for every $x, y \in M$, $fx \neq fy$, where $\alpha \in \mathcal{T}$. Indeed, f and g have a unique common fixed point.*

PROOF. For the necessity, we just follow the proof of Theorem 2.4 for any constant $\alpha \in [0, 1)$. Conversely, if we take an f-iteration $\{fx_n\}_{n=1}^{\infty}$ of x_0 under g, then we can show that $x_n \in M$ for all n, as in the proof of Theorem 2.4. Then the condition of Theorem 3.3 is satisfied.

If $x_0 \notin M$ in Theorem 3.4, then the existence of a common fixed point of f and g follows immediately by putting $x = x_0$ in (1).

In case $f = 1_X$, the above theorem includes Theorem 2 of Rakotch [7]. Note that our proof is simpler.

THEOREM 3.5. *A continuous self-map f of a metric space X has a fixed point iff there exists an f-contractive map g in C_f and a function α in $\mathcal{T}$ such that X is g-orbitally complete w.r.t. f, and*

$$d(gx, gy) \leqq \alpha(x, y)\, d(fx, fy)$$

for all $x, y \in X$, $fx \neq fy$. Indeed

(1) *f and g have a unique common fixed point $\eta \in X$, and*

(2) *for any $x_0 \in X$ and any of its f-iterations $\{fx_n\}_{n=1}^{\infty}$ under g, we have $\lim_n gfx_n = \eta$.*

PROOF. The necessity is clear. For the converse we can apply Theorem 3.3 to any point $x_0 \in X$.

In case $f = 1_X$, for complete X, Theorem 3.5 is due to Geraghty [5]. From Lemma 2.1 and Theorem 3.5, we have

COROLLARY 3.6. *A continuous self-map f of X has a fixed point iff there exists an f-contractive map g in C_f and a function α in $\mathcal{T}$ such that, for some integer $N > 0$, X is g^N-orbitally complete and*

$$d(g^N x, g^N y) \leqq \alpha(x, y)d(fx, fy)$$

for all $x, y \in X$, $fx \neq fy$. Indeed, f and g have a unique common fixed point.

COROLLARY 3.7. *Let f be a bijective continuous self-map of a complete metric space X. If there is an integer $N > 0$ and a function α in $\mathcal{T}$ such that*

$$d(x, y) \leqq \alpha(d(f^N x, f^N y))d(f^N x, f^N y)$$

for every $x, y \in X$, $x \neq y$, then f has a unique fixed point.

A few more generalizations of the Banach contraction principle are obtained from the following.

COROLLARY 3.8. *A continuous self-map f of X has a fixed point iff there exists an f-contractive map g in C_f such that X is g-orbitally complete w.r.t. f and a function $\alpha : (0, \infty) \to [0, 1)$, which satisfies one of the conditions*: (i) *monotone decreasing,* (ii) *monotone increasing,* (iii) *continuous, and* (iv) $\sup_d \alpha(d) < 1$, *such that*

$$d(gx, gy) \leqq \alpha(d(fx, fy))d(fx, fy)$$

for all $x, y \in X$, $fx \neq fy$. Indeed,

(1) *f and g have a unique common fixed point $\eta \in X$, and*

(2) *for any $x_0 \in X$, any f-iteration of x_0 under g converges to some $\zeta \in X$ satisfying $f\zeta = g\zeta = \eta$.*

For $f = 1_X$ and complete X, Corollary 3.8.(i) is due to Rakotch [7], (iii) to Boyd-Wong [2], and (iv) to many authors. For a complete metric space X and a constant α, Corollary 3.8 is due to Jungck [6].

REFERENCES

1. D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. **41** (1966), 101–106.

2. D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

3. Lj. B. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) (N. S.) **12** (26) (1971), 19–26.

4. M. Edelstein, *On fixed and periodic points under contraction mappings*, J. London Math. Soc. **37** (1962), 74–79.

5. M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.

6. G. Jungck, Commuting mappings and fixed points, Amer. Math Monthly **83** (1976), 261–263.

6. E. Rakotch. *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–465.

8. J. S. W. Wong, *Mappings of contractive type on abstract spaces*, J. Math. Anal. Appl. **37** (1972), 331–340.

SEOUL NATIONAL UNIVERSITY, SEOUL, KOREA