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A GENERALIZATION OF A THEOREM OF JANOS AND EDELSTEIN

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ABSTRACT. A generalization of Edelstein's version of a theorem of Janos and its converse are obtained:

THEOREM. *Let X be a compact metrizable topological space, and f, g be continuous self-maps of X such that $gf = fg$ and f is bijective. Then g is injective and $\bigcap_1^\infty g^n X = \{x_0\}$, where $x_0 \in X$, iff, given λ, $0 < \lambda < 1$, a homeomorphism h of X into l_2 exists such that*

$$\|h(gx) - h(gy)\| = \lambda\|h(fx) - h(fy)\|$$

for all $x, y \in X$.

In [2], [3] and [4], Janos obtained the following

THEOREM (JANOS). *Let X be a compact metrizable topological space and f: $X \to X$ a continuous one-to-one mapping with $\bigcap_1^\infty f^n[X]$ a singleton. Given λ, $0 < \lambda < 1$, there exists a metric ρ on X such that the metric topology of (X, ρ) is identical with the original one and $\rho(fx, fy) = \lambda\rho(x, y)$ for all $x, y \in X$.*

M. Edelstein [1] gave a somewhat stronger result as follows:

THEOREM (EDELSTEIN). *Let X be a compact metrizable topological space and f: $X \to X$ a continuous one-to-one mapping with $\bigcap_1^\infty f^n[X] = \{x_0\}$, where $x_0 \in X$. Given λ, $0 < \lambda < 1$, a homeomorphism h of X into l_2 exists such that*

$$\|h(fx') - h(fx'')\| = \lambda\|hx' - hx''\| \quad \text{for all } x', x'' \in X.$$

Note that the converse of Edelstein's theorem holds by the Banach contraction principle.

In this paper we show that a more general result than Edelstein's holds. We need the following result from [5].

THEOREM (JUNGCK). *Let f be a continuous map of a complete metric space (X, d) into itself. Then f has a fixed point in X iff there exist $\alpha \in (0, 1)$ and a map g: $X \to X$ which commutes with f and satisfies $gX \subset fX$ and $d(gx, gy) \leqslant \alpha\, d(fx, fy)$ for all $x, y \in X$. Indeed, f and g have a unique common fixed point.*

Actually, in case $f = 1_X$, the identity map of X, Jungck's theorem implies the Banach principle. Motivated by this fact, we generalize Edelstein's theorem as follows:

THEOREM. *Let X be a compact metrizable topological space, and f, g be*

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continuous self-maps of X such that $gf = fg$ and f is bijective. Then g is injective and $\bigcap_1^\infty g^n X = \{x_0\}$, where $x_0 \in X$, iff, given λ, $0 < \lambda < 1$, a homeomorphism h of X into l_2 exists such that

$$\|h(gx) - h(gy)\| = \lambda \|h(fx) - h(fy)\| \quad \textit{for all } x, y \in X.$$

PROOF. *Necessity*. We may assume $X \setminus \{x_0\} \neq \varnothing$. Let $\mathfrak{B}$ be a countable base for the open nonempty set $X \setminus gX$. For each pair $U, V \in \mathfrak{B}$, such that $\bar{U} \subset V$, there is a continuous $\phi: X \to [0, 1]$ such that $\phi U = 1$ and $\phi(X \setminus V) = 0$. Using the odd positive integers as an index set we obtain a family $\{\phi_{2n-1} | n = 1, 2, \ldots\}$ of maps. Since gX is closed in X and $\phi_{2n-1} fg^{-1}$: $gX \to [0, 1]$ is continuous for each $n = 1, 2, \ldots$, there exists, by the Tietze extension theorem, a continuous $\phi_{2(2n-1)}$: $X \to [0, 1]$ which coincides with $\phi_{2n-1} fg^{-1}$ on gX. Thus $\phi_{2(2n-1)}(gx) = \phi_{2n-1}(fx)$ $(n = 1, 2, \ldots; x \in X)$. Assuming $\phi_{2^{m-1}(2n-1)}$: $X \to [0, 1]$ defined, continuous and satisfying

$$\phi_{2^{m-1}(2n-1)}(gx) = \phi_{2^{m-2}(2n-1)}(fx) \qquad (n = 1, 2, \ldots; x \in X),$$

we define $\phi_{2^m(2n-1)}$: $X \to [0, 1]$ by choosing a continuous extension of $\phi_{2^{m-1}(2n-1)} fg^{-1}$: $gX \to [0, 1]$ to X, thereby obtaining continuous maps $\phi_{2^m(2n-1)}$: $X \to [0, 1]$ for all $m = 1, 2, \ldots; n = 1, 2, \ldots$ and satisfying

$$\phi_{2^m(2n-1)}(gx) = \phi_{2^{m-1}(2n-1)}(fx) \qquad (m = 1, 2, \ldots; n = 1, 2, \ldots; x \in X)$$

and $\phi_{2n-1}(gx) = 0$ $(n = 1, 2, \ldots; x \in X)$.

We now define h as follows: If $k = 2^m(2n - 1)$ we set

$$z_k = \lambda^{m+n} \phi_{2^m(2n-1)}(x) \quad \text{and} \quad hx = (z_1, z_2, \ldots, z_k, \ldots).$$

Obviously $hx \in l_2$. It is easy to see that h is injective and continuous, hence, by the compactness of X, a homeomorphism onto hX. Finally,

$$\begin{aligned}
\|h(gx) - h(gy)\|^2 &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)} \left[\phi_{2^m(2n-1)}(gx) - \phi_{2^m(2n-1)}(gy)\right]^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)} \left[\phi_{2^{m-1}(2n-1)}(fx) - \phi_{2^{m-1}(2n-1)}(fy)\right]^2 \\
&= \lambda^2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)} \left[\phi_{2^m(2n-1)}(fx) - \phi_{2^m(2n-1)}(fy)\right]^2 \\
&= \lambda^2 \|h(fx) - h(fy)\|^2.
\end{aligned}$$

Sufficiency. Since X can be identified with hX, there exist continuous maps $\bar{f}, \bar{g}$: $hX \to hX$ such that $hf = \bar{f}h$ and $hg = \bar{g}h$. Then we have

$$\|\bar{g}(hx) - \bar{g}(hy)\| = \lambda \|\bar{f}(hx) - \bar{f}(hy)\|$$

for all $x, y \in X$. Hence, there is a metric d on X such that $d(gx, gy) = \lambda d(fx, fy)$ for all $x, y \in X$. This shows that g is injective and that f and g have a unique common fixed point x_0 by Jungck's theorem. Since X is bounded, for any $\varepsilon > 0$ there is an $N > 0$ such that $n > N$ implies $\lambda^n \operatorname{diam} X < \varepsilon$. Then

$$d(g^n x, x_0) = d(g^n x, g^n x_0) = \lambda^n d(f^n x, f^n x_0)$$
$$= \lambda^n d(f^n x, x_0) \leq \lambda^n \operatorname{diam} X < \varepsilon,$$

whence we have $\bigcap_1^\infty g^n X = \{x_0\}$.

REMARK. The necessity part of the proof is essentially due to Edelstein. In case $f = 1_X$, our theorem reduces to Edelstein's.

REFERENCES

1. M. Edelstein, *A short proof of a theorem of L. Janos*, Proc. Amer. Math. Soc. **20** (1969), 509–510. MR **38** #2743.

2. L. Janos, *Converse of the Banach theorem in the case of one-to-one contracting mapping*, Notices Amer. Math. Soc. **11** (1964), 686. Abstract #64T-469.

3. ______, *Homothetic property of contractive one-to-one mappings*, Notices Amer. Math. Soc. **13** (1966), 818. Abstract #638-11.

4. ______, *One-to-one contractive mappings on compact space*, Notices Amer. Math. Soc. **14** (1967), 133. Abstract #67T-21.

5. G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

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