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GENERALIZED f-CONTRACTIONS AND FIXED POINT THEOREMS

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1. Introduction. The Banach contraction principle has had numerous generalizations. The following is due to C. S. Wong [8].

THEOREM A. *Let T be a selfmap of a complete metric space (X, d). Then T has a fixed point if there exist selfmaps $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $[0, \infty)$ such that*

(1) $\sum_{i=1}^{5}\alpha_i(t)<t$ *for* $t>0$,

(2) *each* α_i *is upper-semicontinuous from the right,*

(3) $d(Tx, Ty) \leqslant a_1 d(x,y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$

for all pairs of distinct x, y in X, where $a_i = \alpha_i(d(x,y))/d(x,y)$.

This theorem also generalizes some results of D. W. Boyd and J. S. W. Wong [1] and of F. E. Browder [2].

We note that a fixed point of a map $g : X \to X$ is a common fixed point of g and the identity map 1_X of X. Motivated by this fact, G. Jun gck [5] and S. Park [6], [7] obtained some results on fixed points by replacing 1_X by a continuous map $f: X \to X$. In fact, in [5], G. Jungck obtained the following useful extension of the Banach contraction principle.

THEOREM B. *A continuous selfmap f of a complete metric space (X, d) has a fixed point iff there exists $\alpha \in [0, 1)$ and a map $g : X \to X$ which commutes with f and satisfies $gX \subset fX$ and $d(gx, gy) \leqslant \alpha d(fx, fy)$ for all $x, y \in X$. Indeed, f and g have a unique common fixed point ζ.*

In the present paper, we obtain a combined form of Theorems A and B and related results. Consequently, our results extend those in [1], [2], [5] and [8]. Actually our results will be stated for metric spaces more general than complete ones.

2. Generalized f-contractions. Let f be a continuous selfmap of a metric space (X, d), and C_f denote the family of maps $g : X \to X$ such that $gX \subset fX$ and $gf = fg$. Note that C_f is not emptly since f itself belongs to C_f.

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DEFINITION. Given $x\in X$ and a map $g\in C_f$, an *f-iteration of x under g* is a sequence $\{fx_n\}_{n=1}^{\infty}$ given inductively by the rule $fx_n=gx_{n-1}$ for all $n\geqslant 1$, where $x_0=x$.

Note that given $x\in X$, its f-iteration is not unique, however, in case $f=1_X$, an f-iteration of x under g is reduced to the (Picard) sequence of iterates for g.

PROPOSITION 2. 1. *If $x\in X$ has an f-iteration under $g\in C_f$ whose limit is $\eta\in X$, then each $fx_i\in X$ $(i\geqslant 1)$ has an f-iteration whose limit is $f\eta$.*

Proof. Since $ffx_n=fgx_{n-1}=gfx_{n-1}$, $\{ffx_n\}_{n>i}$ is an f-iteration of fx_i under g. From $fx_n\to\eta$ and the continuity of f, we have $ffx_n\to f\eta$.

DEFINITION. A map g in C_f is called a *generalized f-contraction* in the sense of C. S. Wong if there exist functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0,\infty)$ into $[0,\infty)$ such that

(a) each α_i is upper-semicontinuous from the right,

(b) $\sum_{i=1}^{5}\alpha_i(t)<t$, $t>0$,

(c) for any $x, y\in X$, $fx\neq fy$, we have

$$d(gx,gy)\leqslant a_1d(fx,fy)+a_2d(fx,gx)+a_3d(fy,gy)+a_4d(fx,gy)+a_5(fy,gx)$$

where $a_i=\alpha_i(d(fx,fy))/d(fx,fy)$.

Note that if $f=1_X$, the identity map of X, then we get a generalized contraction of C. S. Wong [8].

For a generalized f-contraction, the limit of an f-iteration of x under g depends only on $x\in X$ if exists.

PROPOSITION 2. 2. *Let $g\in C_f$ be a generalized f-contraction. For any $x\in X$, if an f-iteration of x under g has the limit η, so does every convergent f-iteration of x under g.*

Proof. Suppose an f-iteration $\{fx_n\}=\{gx_{n-1}\}_{n=1}^{\infty}$ of $x=x_0\in X$ has limit $\eta\in X$. Let $\{fy_n\}=\{gy_{n-1}\}_{n=1}^{\infty}$ be another convergent f-iteration of $x=y_0$ under g. Then there exists an $i\geqslant 1$ such that $fx_i\neq fy_i$. Hence, we have

$$\begin{aligned} d(\eta,gy_i)&\leqslant d(\eta,gx_i)+d(gx_i,gy_i)\\ &\leqslant d(\eta,gx_i)+a_1d(fx_i,fy_i)+a_2d(fx_i,gx_i)\\ &\quad+a_3d(fy_i,gy_i)+a_4d(fx_i,gy_i)+a_5d(fy_i,gx_i)\end{aligned}$$

and, hence,

$$d(\eta,gy_i)\leqslant a_1d(fx_i,gy_{i-1})+a_4d(fx_i,gy_i)+a_5d(fx_{i+1},gy_{i-1})+o(i)$$

where $\{o(i)\}$ converges to zero. Since $a_1+a_4+a_5<1$, by letting $i\to\infty$, we have $gy_i\to\eta$.

THEOREM 2. 3. *For any generalized f-contraction $g\in C_f$ and for any $x\in X$, there is a Cauchy f-iteration of x under g.*

Proof. Given $x\in X$, we choose an f-iteration $\{fx_n\}=\{gx_{n-1}\}$ of $x=x_0$ under g as follows: If we have $fx_{i+1}=fx_i$ for some i, then we can choose fx_{i+j} by fx_{i+1} for all $j\geqslant 1$ and obtain a Cauchy f-iteration. So we may assume that $d(fx_{n+1}, fx_n)>0$ for each n. Then we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &= d(gx_n, gx_{n-1}) \\ &\leqslant a_1 d(fx_n, fx_{n-1}) + a_2 d(fx_n, gx_n) + a_3 d(fx_{n-1}, gx_{n-1}) \\ &\quad + a_4 d(fx_n, gx_{n-1}) + a_5 d(fx_{n-1}, gx_n) \\ &\leqslant (a_1+a_3+a_5) d(fx_n, fx_{n-1}) + (a_2+a_5) d(fx_n, fx_{n+1}) \end{aligned}$$

and, hence,

$$d(fx_{n+1}, fx_n) \leqslant ((a_1+a_3+a_5)/(1-a_2-a_5)) d(fx_n, fx_{n-1}).$$

By symmetry of x, y in (c), we may assume that $\alpha_5=\alpha_4$. So if we define a function

$$\alpha(t) = [(\alpha_1(t)+\alpha_3(t)+\alpha_5(t))/(t-\alpha_2(t)-\alpha_4(t))]t, \quad t>0,$$

then

$$d(fx_{n+1}, fx_n) \leqslant \alpha(d(fx_n, fx_{n-1}))$$

for all $n\geqslant 1$. Since $\alpha(t)<t$ for $t>0$ by (b), $\{d(fx_{n+1}, fx_n)\}$ is decreasing and converges to some $s\in[0, \infty)$. If $s>0$, then

$$s=\lim_{n\to\infty} d(fx_{n+1}, fx_n) \leqslant \limsup_{n\to\infty} \alpha(d(fx_n, fx_{n-1})).$$

Since α is upper-semicontinuous from the right by (a), we have $s\leqslant\alpha(s)$, a contradiction. So $s=0$. Now we prove that $\{fx_n\}$ is Cauchy. Suppose not. Then there exist $r>0$ and sequences $\{p(n)\}$, $\{q(n)\}$ such that for each n,

$$p(n)>q(n)>n, \quad d(fx_{p(n)}, fx_{q(n)})\geqslant r$$

and by the well-ordering principle

$$d(fx_{p(n)-1}, fx_{q(n)})<r.$$

Then

$$\begin{aligned} r &\leqslant d(fx_{p(n)}, fx_{q(n)}) \\ &\leqslant d(fx_{p(n)-1}, fx_{q(n)}) + d(fx_{p(n)}, fx_{p(n)-1}) \\ &\leqslant r+o(n) \end{aligned}$$

where $o(n)$ converges to 0, hence, $\{d(fx_{p(n)}, fx_{q(n)})\}$ converges to r from the right. By (c),

$$d(fx_{p(n)}, fx_{q(n)})\, d(gx_{p(n)}, gx_{q(n)})$$
$$\leqslant d(fx_{p(n)}, fx_{q(n)})[a_1 d(fx_{p(n)}, fx_{q(n)})$$
$$+a_2 d(fx_{p(n)}, fx_{p(n)+1}) + a_3 d(fx_{q(n)}, fx_{q(n)+1})$$
$$+a_4 d(fx_{p(n)}, fx_{q(n)+1}) + a_5 d(fx_{q(n)}, fx_{p(n)+1})].$$

So by letting $n \to \infty$, we obtain

$$r^2 \leqslant r(a_1 r + a_4 r + a_5 r),$$

a contradiction to (b). Hence $\{fx_n\}$ is Cauchy.

Given a continuous selfmap f of (X, d), we consider a condition on X somewhat more general than completeness.

DEFINITION. Given g in C_f, X is said to be *g-orbitally complete* w. r. t. f if, for any $x \in X$, every Cauchy subsequence of an arbitrary f-iteration $\{fx_n\}_{n=1}^{\infty}$ of x under g converges in X. If X is g-orbitally complete for any $g \in C_f$, then X is said to be *f-complete.*

The g-orbital completeness w. r. t. 1_X is just the g-orbital completeness of Ciric [3], [4].

Clearly, every complete metric space X is f-complete for any f. However, the converse is not true. For example, it may happen that X is not complete, but fX is complete. Then X is f-complete. Note also that every metric space X is g-orbitally complete w. r. t. f if gX is complete.

Combining Proposition 2.2 and Theorem 2.3, we have

COROLLARY 2.4. *Let f be a continuous selfmap of X. If $g \in C_f$ is a generalized f-contraction such that X is g-orbitally complete w. r. t. f, then any $x \in X$ has a convergent f-iteration under g and its limit depends only on x.*

3. Fixed Point Theorems.

Now we have the following main theorem:

THEOREM 3.1. *A continuous selfmap f of a metric space X has a fixed point iff there is a generalized f-contraction g in C_f such that X is g-orbitally complete w. r. t. f. Indeed,*

(1) f and g have a unique common fixed point $\eta \in X$, and

(2) for any $x_0 \in X$ and any f-iteration $\{fx_n\}$ of x_0 under g, we have $\lim_n gfx_n = \eta$.

Proof. Necessity. Suppose that $f\eta = \eta$ for some $\eta \in X$. Define $g: X \to X$ by $gx = \eta$ for all $x \in X$. Then clearly $g \in C_f$. Define $\alpha_1 = (0, \infty) \to [0, \infty)$ by $\alpha_1(t) = \lambda t$ for any $\lambda \in (0, 1)$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. Then (a), (b), and (c) clearly follow.

Sufficiency. Let g be a generalized f-contraction in C_f such that X is g-

orbitally complete w. r. t. f. For any $x=x_0\in X$, we have a convergent f-iteration $\{fx_n\}=\{gx_{n-1}\}$, $n\geqslant 1$, of x, by Corollary 2.4. Let ζ be its limit. Now we show that $f\zeta=g\zeta$. If $fx_{i+1}=fx_i$ for some i, then we could have chosen $fx_{i+1}=fx_{i+2}=\cdots=\zeta$ as in the proof of Theorem 2.3. Hence, $gfx_{i+1}=gfx_{i+2}=\cdots=g\zeta$. Since $gfx_{i+1}=fgx_{i+1}=ffx_{i+2}$, we know that $\{gfx_n\}$ is a tail of an f-iteration of fx_i and, hence, has the limit $f\zeta$, by Proposition 2.1. Therefore, we have $f\zeta=g\zeta$. Suppose that $fx_{n+1}\neq fx_n$ for all $n\geqslant 1$. If $ffx_i\neq f\zeta$ for some $i\geqslant 1$, then

$$\begin{aligned} d(f\zeta, g\zeta) &\leqslant d(f\zeta, fgx_i)+d(gfx_i, g\zeta) \\ &\leqslant d(f\zeta, fgx_i)+a_1d(ffx_i, f\zeta)+a_2d(ffx_i, gfx_i) \\ &\quad +a_3d(f\zeta, g\zeta)+a_4d(ffx_i, g\zeta)+a_5d(f\zeta, gfx_i) \end{aligned}$$

where $a_i=\alpha_i(d(ffx_i, f\zeta))/d(ffx_i, f\zeta)$. Since $ffx_i=fgx_{i-1}=gfx_{i-1}$, letting $i\to\infty$, we obtain

$$d(f\zeta, g\zeta)\leqslant (a_3+a_4)\,d(f\zeta, g\zeta),$$

whence we have $f\zeta=g\zeta$. If $ffx_n=f\zeta$ for all $n\geqslant 1$, we have also $f\zeta=g\zeta$. Suppose now $f\zeta\neq fg\zeta$. Then

$$\begin{aligned} d(f\zeta, fg\zeta) &= d(g\zeta, g^2\zeta) \\ &\leqslant a_1d(f\zeta, fg\zeta)+a_2d(f\zeta, g\zeta)+a_3d(fg\zeta, g^2\zeta) \\ &\quad +a_4d(f\zeta, g^2\zeta)+a_4d(f\zeta, g^2\zeta)+a_5d(fg\zeta, f\zeta) \\ &= (a_1+a_4+a_5)\,d(f\zeta, fg\zeta). \end{aligned}$$

Since $a_1+a_4+a_5<1$, this leads a contradiction. Hence, we must have $f\zeta=fg\zeta$, which shows that $f\zeta=g\zeta$ is a common fixed point of f and g. If f and g have two common fixed point α, β in X, $\alpha\neq\beta$, then

$$\begin{aligned} d(\alpha, \beta) &= d(g\alpha, g\beta) \\ &\leqslant a_1d(\alpha, \beta)+a_4d(\alpha, \beta)+a_5d(\alpha, \beta)<d(\alpha, \beta), \end{aligned}$$

which leads a contradiction. Hence f and g have a unique common fixed point.

In case $f=1_X$, a generalized contraction g has a unique fixed point by Theorem 3.1. This is just Theorem 1 of C. S. Wong [8], which is reduced to a result of Boyd-Wong [1]. If $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and α_1 is a constant in [0, 1), Theorem 3.1 is reduced to a result of Jungck [5]. In fact, a number of known fixed point theorems are consequences of Theorem 3.1.

In Theorem 3.1, it can be proved that if X is compact and if "$<$" in (b) is interchanged with "$\leqslant$" in (c), then it is still true.

CORROLLARY 3.2. *Let f be a continuous selfmap of a complete metric space*

(X, d). *If* $g \in C_f$ *and if there is a positive integer* k *such that* g^k *is a generalized* f*-contraction, then* f *and* g *have a unique common fixed point.*

Proof. Clearly $g^k \in C_f$ and by Theorem 3.1, there is a unique $\eta \in X$ such that $\eta = f\eta = g^k\eta$. But then, since f and g commute, we can write $g\eta = f(g\eta) = g^k(g\eta)$, which says that $g\eta$ is also a common fixed point of f and g^k. The uniqueness implies $\eta = g\eta = f\eta$.

THEOREM 3.3. *Let* f *be a continuous selfmap of a metric space* X *and* g *be a generalized* f*-contraction in* C_f *such that* X *is* g*-orbitally complete w.r.t.* f *as in* Theorem 3.1. *Suppose further that each* α_i *is increasing. Then*

(i) $d(gfx_n, \eta) \leqslant \alpha^n(d(gfx, \eta))$ *for all* $n \geqslant 0$ *where* η *is the common fixed point of* f *and* g, $\alpha(0) = 0$, *and for* $t > 0$,

$$\alpha(t) = [(2\alpha_1(t) + \textstyle\sum_{i=2}^{5} \alpha_i(t)) / (2t - \sum_{i=2}^{5} \alpha_i(t))]t;$$

(ii) α *is increasing, continuous from the right and for any* $t \in [0, \infty)$, $\{\alpha^n(t)\}$ *converges to* 0.

Hence, $\{gfx_n\}$ *converges uniformly to the common fixed point of* f *and* g *on any bounded subset of* X.

Proof. (i) Let $x \in X$ be such that $gfx \neq \eta$ and $b_n = d(gfx_n, \eta)$ for $n \geqslant 0$. By (c), we have

$$\begin{aligned} b_0 b_1 = b_0 d(gfx_1, \eta) &= b_0 d(g^2x, g\eta) \\ &\leqslant \alpha_1(b_0) d(fgx, \eta) + \alpha_2(b_0) d(fgx, g^2x) \\ &\qquad + \alpha_4(b_0) d(fgx, \eta) + \alpha_5(b_0) d(\eta, g^2x) \\ &\leqslant \alpha_1(b_0) b_0 + \alpha_2(b_0)(b_0 + b_1) + \alpha_4(b_0) b_0 + \alpha_5(b_0) b_1 \end{aligned}$$

and, hence,

$$b_1 \leqslant [(\alpha_1(b_0) + \alpha_2(b_0) + \alpha_4(b_0)) / (b_0 - \alpha_2(b_0) - \alpha_5(b_0))] b_0.$$

Similarly

$$b_2 \leqslant [(\alpha_1(b_1) + \alpha_2(b_1) + \alpha_4(b_1)) / (b_1 - \alpha_2(b_1) - \alpha_5(b_1))] b_1.$$

Because of the symmetry of x, y, (c) still holds if we replace $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ respectively by

$$\alpha_1, \ (\alpha_2 + \alpha_3)/2, \ (\alpha_2 + \alpha_3)/2, \ (\alpha_4 + \alpha_5)/2, \ (\alpha_4 + \alpha_5)/2.$$

Thus $b_1 \leqslant \alpha(b_0)$, $b_2 \leqslant \alpha(b_1)$, and by induction $b_{n+1} \leqslant \alpha(b_n)$, $n \geqslant 0$. Since α is increasing, by induction we have

$$d(gfx_n, \eta) = b_n \leqslant \alpha^n(b_0) = \alpha^n(d(gfx, x)), \quad n \geqslant 0.$$

(ii) Each α_i is increasing and continuous from the right, so is α. Let $t > 0$. By (b), $\alpha(t) < t$. So $\{\alpha^n(t)\}$ is decreasing and converges to some $t_0 \in$

$[0, \infty)$. Suppose $t_0>0$. Then by the right continuity of α,

$$t_0=\lim_{n\to\infty}\alpha^{n+1}(t)\leqslant\alpha(\lim_{n\to\infty}\alpha^n(t))=\alpha(t_0),$$

which is a contradiction to $\alpha(t)<t$ for $t>0$. Hence, $t_0=0$.

In case $f=1_X$, Theorem 3.3 is due to C. S. Wong [8]. Note that our proof is a slight modification of his. When $f=1_X$, $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and X is bounded, Theorem 3.3 is reduced to a result of F. E. Browder [2, Theorem 1].

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