

Mappings Generating Upper-semicontinuous Decompositions of Spaces with Coherent Topologies

SEHIE PARK

1. Introduction

In this paper we generalize the concepts of compactly generated spaces (or Hausdorff k -spaces) and reflexive compact mappings. Using these concepts we obtain sufficient conditions for mappings to generate upper-semicontinuous (u.s.c.) decompositions of certain types of spaces with coherent topologies. We also show that some of the results generalize similar situations which are given previously.

2. Terminologies

A k -space X is a topological space having a topology coherent with the collection of its closed compact subsets; i.e., if a subset A of X intersects each closed compact set in a closed set, then A is closed [4]. A Hausdorff k -space is said to be *compactly generated* [6]. A topological space X is said to be *locally paracompact* if every point of X has a closed neighborhood which is a paracompact subspace of X [2]. Throughout this paper X and Y will represent topological spaces and f will be a mapping (continuous function) of X into Y . A mapping f of X into Y is said to be *reflexive compact* provided that $f^{-1}f(C)$ is compact for every compact subset C of X [3]. All terminologies given above will be generalized in Sections 3 and 4. A mapping f of X into Y is said to *generate an u.s.c. decomposition* of X if for every open set U of X , the union V of point inverses $f^{-1}(y)$ contained in U is an open subset of X .

3. P -generated spaces

Throughout this paper, a topological property P is said to be *admissible* if it is inherited by closed sets.

3.1. DEFINITION. Let X be a space and P an admissible property. A P -set in X is a closed subset of X which possesses the property P . X is said to be *P -generated* if it has a topology coherent with the collection of its P -sets; i.e., if a subset A of X intersects each P -set in a closed set, then A is closed.

3.2. DEFINITION. Let P and Q be admissible properties. We shall write $P \subset Q$ if and only if every P -set is a Q -set.

For instance, let P and Q be the compactness and the paracompactness, respectively. Then $P \subset Q$ in a Hausdorff space.

3.3. DEFINITION. A neighborhood which is a P -set will be called P -neighborhood. A space X is said to be *locally P* provided that every point of X has a P -neighborhood.

It is obvious that if $P \subset Q$ and X is locally P , then X is locally Q .

3.4. PROPOSITION. *A locally P -space is P -generated.*

Proof. Let B be a non-closed subset of X and suppose x is an accumulation point of B which does not belong to B . Since X is locally P there is a P -neighborhood U of x and the intersection $B \cap U$ is not closed because x is an accumulation point but not a member of $B \cap U$.

3.5. COROLLARY. *A locally compact Hausdorff space is compactly generated [4].*

3.6. PROPOSITION. *If a space X is P -generated and $P \subset Q$, then X is Q -generated.*

Proof. Let B be a non-closed subset of X . If X is P -generated there is a P -set V of X such that $V \cap B$ is not closed. Since $P \subset Q$, V is also Q -set.

We can define an analogue to compactly generated spaces.

3.7. DEFINITION. A *paracompactly generated space* is a Hausdorff space having a topology coherent with the collection of its closed paracompact subsets.

From Propositions 3.4 and 3.6 we obtain

3.8. COROLLARY. *A Hausdorff space which is either locally paracompact or compactly generated is paracompactly generated.*

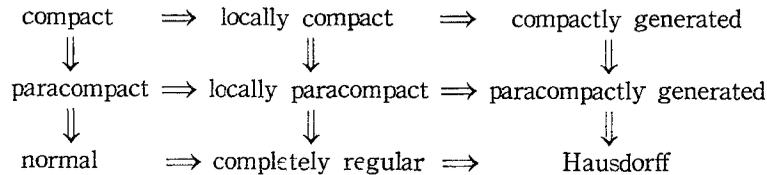
It is established that a Hausdorff space which satisfies one of the following conditions is compactly generated.

- (a) locally compact [4]
- (b) the first axiom of countability [4]
- (c) product of a compactly generated space and a locally compact Hausdorff space [1, 6]

Hence, from the latter part of Corollary 3.8, we obtain

3.9. COROLLARY. *A Hausdorff space which satisfies one of the conditions (a), (b) and (c) is paracompactly generated.*

In summary we have



in the class of Hausdorff spaces and we know none of the arrows can be reversed.

Especially, the following example in [1] which shows the product of two compactly generated spaces need not be a compactly generated space serves as a paracompactly generated space which is not compactly generated.

3.10. EXAMPLE. Let X be the dual space of an infinite dimensional Fréchet space with the compact-open topology is compactly generated space which is not locally compact. $F = C(X, [0, 1])$ with the compact-open topology is metrizable. However $X \times F$ is not compactly generated. It follows from [5, Prop. 4] that $X \times F$ is paracompact which implies it is paracompactly generated.

4. Reflexive P -mappings

4.1. DEFINITION. Let f be a mapping of X into Y . f is called a *reflexive P -mapping* provided that $f^{-1}f(P)$ is a P -set for every P -set P of X .

With the above definition we obtain a sufficient condition for mappings to generate u.s.c. decompositions of P -generated spaces.

4.2. THEOREM. *Let X be a P -generated space and f a mapping of X into Y . If f is a reflexive P -mapping then f generates an upper-semicontinuous decomposition of X .*

This is a generalization of [3, Theorem 1] and the following proof is a slight modification of that in [3].

Proof. Let U be an open set in X containing a point inverse and V the union of the point inverses contained in U . We show that $\bar{u} - V$ is closed in X which in turn implies V is open in X and consequently f generates an u.s.c. decomposition. Let P be a P -set in X such that $H = (\bar{U} - V) \cap P \neq \phi$. For each point x in H , $f^{-1}f(x) \cap (X - U) \neq \phi$. Thus we obtain

$$f^{-1}f(H) \cap (X - U) = f^{-1}f(P \cap \bar{U}) \cap (X - U). \quad (1)$$

The right member is a P -set so is the left member. Denoting the set in (1) above by M we obtain

$$f^{-1}f(M) \cap \bar{U} = f^{-1}f(P \cap \bar{U}) \cap (\bar{U} - V).$$

The left member is a P -set so is the right member. The set $P \cap (\bar{U} - V)$ is closed for it is the intersection of the P -sets P and $f^{-1}f(P \cap \bar{U}) \cap (\bar{U} - V)$. Thus X being P -generated implies $\bar{U} - V$ is a closed subset of X .

Since the closedness is admissible and every closed mapping is reflexive closed, we obtain

4.3. COROLLARY. *Let f be a mapping of X into Y . If f is closed then f generates an upper-semicontinuous decomposition of X .*

It is well-known the converse of Corollary 4.3 also holds.

4.4. COROLLARY. *Let X be a compactly generated space and f a mapping of X into Y . If f is reflexive compact, then f generates an upper-semicontinuous decomposition of X [3, Theorem 1].*

4.5. COROLLARY. *Let X be a paracompactly generated space and f a mapping of X into Y . If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X .*

Combining Corollary 3.8 with Corollary 4.5 we obtain the following corollaries.

4.6. COROLLARY. *Let X be a compactly generated space and f a mapping of X into Y . If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X .*

4.7. COROLLARY. *Let X be a locally paracompact Hausdorff space and f a mapping of X into Y . If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X .*

References

1. R. W. Bagley and J. S. Yang, *On k -spaces and function spaces*, Proc. Amer. Math. Soc. **17** (1966), 703–705.
2. N. Bourbaki, *General Topology*, Addison-Wesley, Reading, 1966.
3. Edwin Duda, *Reflexive compact mappings*, Proc. Amer. Math. Soc. **17**(1966), 688–693.
4. J. L. Kelly, *General Topology*, Van Nostrand, New York, 1963.
5. E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc. **4**(1953), 831–838.
6. E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

Seoul National University